Configurations in abelian categories. IV. Invariants and changing stability conditions

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Abstract

This is the last in a series on *configurations* in an abelian category $\mathcal{A}$. Given a finite poset $(I,\preceq)$, an $(I,\preceq)$-configuration (σ,ι,π) is a finite collection of objects $\sigma(J)$ and morphisms $\iota(J,K)$ or $\pi(J,K):\sigma(J)\rightarrow\sigma(K)$ in $\mathcal{A}$ satisfying some axioms, where J,K are subsets of I. Configurations describe how an object X in $\mathcal{A}$ decomposes into subobjects.

The first paper defined configurations and studied moduli spaces of configurations in $\mathcal{A}$, using Artin stacks. It showed well-behaved moduli stacks $\mathfrak{Obj}_{\mathcal{A}},\mathfrak{M}(I,\preceq)_{\mathcal{A}}$ of objects and configurations in $\mathcal{A}$ exist when $\mathcal{A}$ is the abelian category $\mathrm{coh}(P)$ of coherent sheaves on a projective scheme P, or mod-$\mathbb{K}Q$ of representations of a quiver Q. The second studied algebras of *constructible functions* and *stack functions* on $\mathfrak{Obj}_{\mathcal{A}}$.

The third introduced *stability conditions* $(\tau,T,\leqslant)$ on $\mathcal{A}$, and showed the moduli space $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ of τ-semistable objects in class α is a constructible subset in $\mathfrak{Obj}_{\mathcal{A}}$, so its characteristic function $\delta_{\mathrm{ss}}^{\alpha}(\tau)$ is a constructible function. It formed algebras $\mathcal{H}_{\tau}^{\mathrm{pa}},\mathcal{H}_{\tau}^{\mathrm{to}},\bar{\mathcal{H}}_{\tau}^{\mathrm{pa}},\bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ of constructible and stack functions on $\mathfrak{Obj}_{\mathcal{A}}$, and proved many identities in them.

In this paper, if $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$ are stability conditions on $\mathcal{A}$ we write $\delta_{\mathrm{ss}}^{\alpha}(\tilde\tau)$ in terms of the $\delta_{\mathrm{ss}}^{\beta}(\tau)$, and deduce the algebras $\mathcal{H}_{\tau}^{\mathrm{pa}},\ldots,\bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ are independent of $(\tau,T,\leqslant)$. We study *invariants* $I_{\mathrm{ss}}^{\alpha}(\tau)$ or $I_{\mathrm{ss}}(I,\preceq,\kappa,\tau)$ 'counting' τ-semistable objects or configurations in $\mathcal{A}$, which satisfy additive and multiplicative identities. We compute them completely when $\mathcal{A}=\text{mod-}\mathbb{K}Q$ or $\mathcal{A}=\mathrm{coh}(P)$ for P a smooth curve. We also find invariants with special properties when $\mathcal{A}=\mathrm{coh}(P)$ for P a smooth surface with K_P^{-1} nef, or a Calabi–Yau 3-fold.

1 Introduction

This is the fourth in a series of papers [29–31] on *configurations*. Given an abelian category $\mathcal{A}$ and a finite partially ordered set (poset) $(I,\preceq)$, we define an $(I,\preceq)$-*configuration* (σ,ι,π) in $\mathcal{A}$ to be a collection of objects $\sigma(J)$ and morphisms $\iota(J,K)$ or $\pi(J,K):\sigma(J)\rightarrow\sigma(K)$ in $\mathcal{A}$ satisfying certain axioms, for $J,K\subseteq I$.

The first paper [29] defined configurations, developed their basic properties, and studied moduli spaces of configurations in $\mathcal{A}$, using the theory of Artin stacks. It proved well-behaved moduli stacks $\mathfrak{Obj}_{\mathcal{A}},\mathfrak{M}(I,\preceq)_{\mathcal{A}}$ of objects and

configurations exist when $\mathcal{A}$ is the abelian category $\mathrm{coh}(P)$ of coherent sheaves on a projective $\mathbb{K}$-scheme P, or mod-$\mathbb{K}Q$ of representations of a quiver Q. The second [30] studied algebras of *constructible functions* $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and *stack functions* $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ on $\mathfrak{Obj}_{\mathcal{A}}$, motivated by *Ringel–Hall algebras*.

The third paper [31] studied (*weak*) *stability conditions* $(\tau, T, \leqslant)$ on $\mathcal{A}$, which include *slope stability* on mod-$\mathbb{K}Q$ and *Gieseker stability* on $\mathrm{coh}(P)$. If $(\tau, T, \leqslant)$ is *permissible* then the moduli space $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ of τ-semistable objects X in $\mathcal{A}$ with $[X] = \alpha$ in $K(\mathcal{A})$ is a *constructible set* in $\mathfrak{Obj}_{\mathcal{A}}$, so its characteristic function $\delta_{\mathrm{ss}}^{\alpha}(\tau)$ is a *constructible function*. We used this to define interesting algebras $\mathcal{H}_{\tau}^{\mathrm{pa}}, \mathcal{H}_{\tau}^{\mathrm{to}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ and Lie algebras $\mathcal{L}_{\tau}^{\mathrm{pa}}, \mathcal{L}_{\tau}^{\mathrm{to}}, \bar{\mathcal{L}}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{L}}_{\tau}^{\mathrm{to}}$ in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, and prove many identities in them.

The first goal of this paper is to understand *how moduli spaces* $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ *transform as we change the* (*weak*) *stability condition* $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$. We express this as the following identity in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$, which is equation (44) below:

$$\delta_{\mathrm{ss}}^{\alpha}(\tilde{\tau}) = \sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde{\tau})\cdot \\ &\delta_{\mathrm{ss}}^{\kappa(1)}(\tau) * \delta_{\mathrm{ss}}^{\kappa(2)}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\kappa(n)}(\tau). \end{aligned} \tag{1}$$

Here $S(*, \tau, \tilde{\tau})$ are *explicit combinatorial coefficients* equal to 1,0 or -1, and '$*$' is the associative, noncommutative multiplication on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ studied in [30].

Roughly speaking, (1) characterizes whether $X \in \mathcal{A}$ is $\tilde{\tau}$-semistable in terms of τ-semistability, via an *inclusion-exclusion process* upon *filtrations* $0 = A_0 \subset \cdots \subset A_n = X$ with τ-semistable factors $S_i = A_i/A_{i-1}$. Writing $\kappa(i) = [S_i]$ in $K(\mathcal{A})$, the coefficient $S(\cdots)$ depends on the orderings of $\tau \circ \kappa(i)$ and $\tilde{\tau} \circ \kappa(i)$ for $i = 1, \ldots, n$ in the total orders $(T, \leqslant)$ and $(\tilde{T}, \leqslant)$, and this determines whether a filtration is included, if $S(\cdots) = 1$, or excluded, if $S(\cdots) = -1$.

We say that $(\tilde{\tau}, \tilde{T}, \leqslant)$ *dominates* $(\tau, T, \leqslant)$ if $\tau(\alpha) \leqslant \tau(\beta)$ implies $\tilde{\tau}(\alpha) \leqslant \tilde{\tau}(\beta)$ for all $\alpha, \beta \in C(\mathcal{A})$. In this case (1) follows easily from the facts that each $X \in \mathcal{A}$ has a unique *Harder–Narasimhan filtration* $0 = A_0 \subset \cdots \subset A_n = X$ with $S_i = A_i/A_{i-1}$ τ-semistable and $\tau([S_1]) > \cdots > \tau([S_n])$, and then X is $\tilde{\tau}$-semistable if and only if $\tilde{\tau}([S_i]) = \tilde{\tau}(\alpha)$ for all i. For the general case we go via a weak stability condition $(\hat{\tau}, \hat{T}, \leqslant)$ dominating both $(\tau, T, \leqslant)$ and $(\tilde{\tau}, \tilde{T}, \leqslant)$. For $\mathcal{A} = \mathrm{coh}(P)$ equation (1) may have infinitely many nonzero terms, and *converges* in a suitable sense.

The second goal of the paper is to study *systems of invariants* of $\mathcal{A}$ and $(\tau, T, \leqslant)$ which 'count' τ-semistable objects or configurations in $\mathcal{A}$. Obviously there are many ways of doing this, so we need to decide what are the most interesting, or useful, ways to define invariants. The point of view we take is that the invariants are interesting if they satisfy *natural identities*, and the more identities the better. Such identities are powerful tools for calculating the invariants in examples, as we shall see.

We obtain our invariants $I_{\mathrm{ss}}^{\alpha}(\tau)$ by applying some invariant Υ of constructible sets in Artin stacks to the moduli spaces $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$. If Υ takes values in a vector space Λ and $\Upsilon(S \cup T) = \Upsilon(S) + \Upsilon(T)$ when S, T are constructible sets with $S \cap T = \emptyset$, then constructible function identities such as (1) and many more in [31] translate to *additive identities* on the invariants, such as transformation

laws under change of (weak) stability condition. This is our basic assumption, and holds for Euler characteristics, virtual Poincaré polynomials, and so on.

If also Λ is a commutative algebra and Υ has multiplicative properties such as $\Upsilon(S\times T)=\Upsilon(S)\Upsilon(T)$ or $\Upsilon([X/G])=\Upsilon(X)\Upsilon(G)^{-1}$ then we can derive extra *multiplicative identities* on the $I_{\rm ss}^\alpha(\tau)$. Usually these multiplicative identities require extra conditions on the groups $\mathrm{Ext}^i(X,Y)$ for $X,Y\in\mathcal{A}$ and $i>1$, and can be interpreted in terms of morphisms from a stack (Lie) algebra in $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ to some explicit algebra, as in [30, §6]. These assumptions on $\mathrm{Ext}^i(X,Y)$ mean that our invariants have good properties in special cases which we focus on, namely, when $\mathcal{A}=\text{mod-}\mathbb{K}Q$, or $\mathcal{A}=\mathrm{coh}(P)$ for P a smooth curve, or P a smooth surface with K_P^{-1} nef, or P a Calabi–Yau 3-fold.

Here is an overview of the paper. Section 2 gives background material on Artin stacks, constructible functions and stack functions from [27, 28], and §3 briefly reviews the first three papers [29–31]. Section 4 defines and studies the *transformation coefficients* $S(*,\tau,\tilde\tau)$ appearing in (1), and related coefficients $T,U(*,\tau,\tilde\tau)$; this part of the paper is wholly *combinatorial*.

In §5 we prove (1), its stack function analogue, and transformation laws for other families of constructible and stack functions $\delta_{\rm ss},\bar\delta_{\rm ss}(I,\preceq,\kappa,\tau)$ and $\epsilon^\alpha,\bar\epsilon^\alpha(\tau)$. When $\mathcal{A}=\text{mod-}\mathbb{K}Q$ equation (1) has only finitely many terms, but when $\mathcal{A}=\mathrm{coh}(P)$ for P a projective $\mathbb{K}$-scheme it might have infinitely many nonzero terms, and holds with an appropriate notion of convergence. We show there are only finitely many nonzero terms in (1) if P is a smooth surface.

Section 6 studies some families of invariants $I_{\rm ss}(I,\preceq,\kappa,\tau)^\Lambda$, $I_{\rm ss}^\alpha(\tau)^\Lambda$, $J^\alpha(\tau)^{\Lambda^\circ}$, $J^\alpha(\tau)^\Omega$ taking values in $\mathbb{Q}$-algebras $\Lambda,\Lambda^\circ,\Omega$ which 'count' τ-semistable objects or configurations in $\mathcal{A}$. We determine their transformation laws under change of stability condition, and additive and multiplicative identities they satisfy under conditions on $\mathrm{Ext}^i(X,Y)$ for $i>1$ and $X,Y\in\mathcal{A}$.

We compute the invariants completely when $\mathcal{A}=\text{mod-}\mathbb{K}Q$ or $\mathcal{A}=\mathrm{coh}(P)$ for P a smooth curve, recovering results of Reineke and Harder–Narasimhan–Atiyah–Bott. We also find invariants with special multiplicative transformation laws when $\mathcal{A}=\mathrm{coh}(P)$ for P a smooth surface with K_P^{-1} nef or a Calabi–Yau 3-fold. For surfaces P with $c_1(P)=0$, such as $K3$ surfaces, we define invariants $\bar J^\alpha(\tau)^\Lambda$ which are independent of the choice of Gieseker stability condition $(\tau,T,\leqslant)$ on P. We discuss the connection of our invariants with *Donaldson invariants* of surfaces and *Donaldson–Thomas invariants* of Calabi–Yau 3-folds, and make some conjectures on the existence of invariants combining good features of the various sets of invariants.

Finally, §7 suggests problems for future research: extending the whole programme to triangulated categories, combining the invariants in generating functions, with applications to Mirror Symmetry, and use of esoteric kinds of stacks to weaken the assumptions we need on $\mathrm{Ext}^i(X,Y)$.

A sequel [32] explains how to encode some of the invariants we study into *holomorphic generating functions* on the complex manifold of stability conditions. These satisfy an interesting p.d.e., that can be interpreted as the flatness of a connection. This will be discussed in §7.

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2 Background material

We begin with some background material on Artin stacks, constructible functions, 'stack functions', and motivic invariants. Sections 2.1–2.3 are drawn from [27, 28], and §2.4 is new.

2.1 Artin $\mathbb{K}$-stacks and constructible functions

Let $\mathbb{K}$ be an algebraically closed field. There are four main classes of 'spaces' over $\mathbb{K}$ used in algebraic geometry, in increasing order of generality:

$$\mathbb{K}\text{-varieties} \subset \mathbb{K}\text{-schemes} \subset \text{algebraic } \mathbb{K}\text{-spaces} \subset \text{algebraic } \mathbb{K}\text{-stacks}.$$

Algebraic stacks (also known as Artin stacks) were introduced by Artin, generalizing *Deligne–Mumford stacks*. For a good introduction to algebraic stacks see Gómez [20], and for a thorough treatment see Laumon and Moret-Bailly [35]. We make the convention that all algebraic $\mathbb{K}$-stacks in this paper are *locally of finite type*, and $\mathbb{K}$-substacks are *locally closed*.

We define the set of $\mathbb{K}$-*points* of a stack.

Definition 2.1. Let $\mathfrak{F}$ be a $\mathbb{K}$-stack. Write $\mathfrak{F}(\mathbb{K})$ for the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$. Elements of $\mathfrak{F}(\mathbb{K})$ are called $\mathbb{K}$-*points*, or *geometric points*, of $\mathfrak{F}$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism then composition with ϕ induces a map of sets $\phi_* : \mathfrak{F}(\mathbb{K}) \to \mathfrak{G}(\mathbb{K})$.

For a 1-morphism $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$, the *stabilizer group* $\operatorname{Iso}_{\mathbb{K}}(x)$ is the group of 2-morphisms $x \to x$. When $\mathfrak{F}$ is an algebraic $\mathbb{K}$-stack, $\operatorname{Iso}_{\mathbb{K}}(x)$ is an *algebraic* $\mathbb{K}$*-group*. We say that $\mathfrak{F}$ *has affine geometric stabilizers* if $\operatorname{Iso}_{\mathbb{K}}(x)$ is an affine algebraic $\mathbb{K}$-group for all 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$.

As an algebraic $\mathbb{K}$-group up to isomorphism, $\operatorname{Iso}_{\mathbb{K}}(x)$ depends only on the isomorphism class $[x] \in \mathfrak{F}(\mathbb{K})$ of x in $\operatorname{Hom}(\operatorname{Spec}\mathbb{K}, \mathfrak{F})$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism, composition induces a morphism of algebraic $\mathbb{K}$-groups $\phi_* : \operatorname{Iso}_{\mathbb{K}}([x]) \to \operatorname{Iso}_{\mathbb{K}}\bigl(\phi_*([x])\bigr)$, for $[x] \in \mathfrak{F}(\mathbb{K})$.

The theory of *constructible functions* on $\mathbb{K}$-stacks was developed in [27].

Definition 2.2. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack. We call $C \subseteq \mathfrak{F}(\mathbb{K})$ *constructible* if $C = \bigcup_{i \in I} \mathfrak{F}_i(\mathbb{K})$, where $\{\mathfrak{F}_i : i \in I\}$ is a finite collection of finite type algebraic $\mathbb{K}$-substacks $\mathfrak{F}_i$ of $\mathfrak{F}$. We call $S \subseteq \mathfrak{F}(\mathbb{K})$ *locally constructible* if $S \cap C$ is constructible for all constructible $C \subseteq \mathfrak{F}(\mathbb{K})$.

A function $f : \mathfrak{F}(\mathbb{K}) \to \mathbb{Q}$ is called *constructible* if $f(\mathfrak{F}(\mathbb{K}))$ is finite and $f^{-1}(c)$ is a constructible set in $\mathfrak{F}(\mathbb{K})$ for each $c \in f(\mathfrak{F}(\mathbb{K})) \setminus \{0\}$. A function $f : \mathfrak{F}(\mathbb{K}) \to \mathbb{Q}$ is called *locally constructible* if $f \cdot \delta_C$ is constructible for all

constructible $C \subseteq \mathfrak{F}(\mathbb{K})$, where δ_C is the characteristic function of C. Write $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{LCF}(\mathfrak{F})$ for the $\mathbb{Q}$-vector spaces of $\mathbb{Q}$-valued constructible and locally constructible functions on $\mathfrak{F}$.

Following [27, Def.s 4.8, 5.1 & 5.5] we define *pushforwards* and *pullbacks* of constructible functions along 1-morphisms. We need $\operatorname{char}\mathbb{K} = 0$.

Definition 2.3. Let $\mathbb{K}$ have *characteristic zero* and $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $C \subseteq \mathfrak{F}(\mathbb{K})$ be constructible. Then [27, Def. 4.8] defines the *naïve Euler characteristic* $\chi^{\mathrm{na}}(C)$ of C. It is called *naïve* as it takes no account of stabilizer groups. For $f \in \mathrm{CF}(\mathfrak{F})$, define

$$\chi^{\mathrm{na}}(\mathfrak{F}, f) = \textstyle\sum_{c \in f(\mathfrak{F}(\mathbb{K}))\setminus\{0\}} c\,\chi^{\mathrm{na}}\bigl(f^{-1}(c)\bigr) \quad \text{in } \mathbb{Q}.$$

Let $\mathfrak{F}, \mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$ a representable 1-morphism. Then for any $x \in \mathfrak{F}(\mathbb{K})$ we have an injective morphism $\phi_* : \mathrm{Iso}_{\mathbb{K}}(x) \rightarrow \mathrm{Iso}_{\mathbb{K}}\bigl(\phi_*(x)\bigr)$ of affine algebraic $\mathbb{K}$-groups. Define $m_\phi : \mathfrak{F}(\mathbb{K}) \rightarrow \mathbb{Z}$ by $m_\phi(x) = \chi\bigl(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))\bigr)$. For f in $\mathrm{CF}(\mathfrak{F})$, define $\mathrm{CF}^{\mathrm{stk}}(\phi)f : \mathfrak{G}(\mathbb{K}) \rightarrow \mathbb{Q}$ by $\mathrm{CF}^{\mathrm{stk}}(\phi)f(y) = \chi^{\mathrm{na}}\bigl(\mathfrak{F}, m_\phi \cdot f \cdot \delta_{\phi_*^{-1}(y)}\bigr)$ for y in $\mathfrak{G}(\mathbb{K})$, where $\delta_{\phi_*^{-1}(y)}$ is the characteristic function of $\phi_*^{-1}(\{y\}) \subseteq \mathfrak{G}(\mathbb{K})$. Then $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{F}) \rightarrow \mathrm{CF}(\mathfrak{G})$ is a $\mathbb{Q}$-linear map called the *stack pushforward*.

Let $\theta : \mathfrak{F} \rightarrow \mathfrak{G}$ be a finite type 1-morphism. If $C \subseteq \mathfrak{G}(\mathbb{K})$ is constructible then so is $\theta_*^{-1}(C) \subseteq \mathfrak{F}(\mathbb{K})$. It follows that if $f \in \mathrm{CF}(\mathfrak{G})$ then $f \circ \theta_*$ lies in $\mathrm{CF}(\mathfrak{F})$. Define the *pullback* $\theta^* : \mathrm{CF}(\mathfrak{G}) \rightarrow \mathrm{CF}(\mathfrak{F})$ by $\theta^*(f) = f \circ \theta_*$. It is a linear map.

Here [27, Th.s 5.4, 5.6 & Def. 5.5] are some properties of these.

Theorem 2.4. *Let $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\beta : \mathfrak{F} \rightarrow \mathfrak{G}$, $\gamma : \mathfrak{G} \rightarrow \mathfrak{H}$ be 1-morphisms. Then*

$$\mathrm{CF}^{\mathrm{stk}}(\gamma \circ \beta) = \mathrm{CF}^{\mathrm{stk}}(\gamma) \circ \mathrm{CF}^{\mathrm{stk}}(\beta) : \mathrm{CF}(\mathfrak{F}) \rightarrow \mathrm{CF}(\mathfrak{H}), \tag{2}$$

$$(\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \mathrm{CF}(\mathfrak{H}) \rightarrow \mathrm{CF}(\mathfrak{F}), \tag{3}$$

supposing β, γ representable in (2)*, and of finite type in* (3)*. If*

$$\begin{tikzcd} \mathfrak{E} \arrow[r, "\eta"] \arrow[d, "\theta"] & \mathfrak{G} \arrow[d, "\psi"] \\ \mathfrak{F} \arrow[r, "\phi"] & \mathfrak{H} \end{tikzcd}$$

is a Cartesian square with η, ϕ representable and θ, ψ of finite type, then the following commutes:

$$\begin{tikzcd} \mathrm{CF}(\mathfrak{E}) \arrow[r, "\mathrm{CF}^{\mathrm{stk}}(\eta)"'] & \mathrm{CF}(\mathfrak{G}) \\ \mathrm{CF}(\mathfrak{F}) \arrow[u, "\theta^*"'] \arrow[r, "\mathrm{CF}^{\mathrm{stk}}(\phi)"] & \mathrm{CF}(\mathfrak{H}). \arrow[u, "\psi^*"] \end{tikzcd} \tag{4}$$

As discussed in [27, §3.3] for the $\mathbb{K}$-scheme case, equation (2) is *false* for algebraically closed fields $\mathbb{K}$ of characteristic $p > 0$. In [27, §5.3] we extend Definition 2.3 and Theorem 2.4 to *locally constructible functions* $\mathrm{LCF}(\mathfrak{F})$. The main differences are in which 1-morphisms must be of *finite type*.

2.2 Stack functions

Stack functions are a universal generalization of constructible functions introduced in [28]. Here [28, Def. 3.1] is the basic definition. Throughout $\mathbb{K}$ is algebraically closed of arbitrary characteristic, except when we specify $\operatorname{char}\mathbb{K} = 0$.

Definition 2.5. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $(\mathfrak{R}, \rho)$, where $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\rho : \mathfrak{R} \to \mathfrak{F}$ is a representable 1-morphism. We call two pairs $(\mathfrak{R}, \rho)$, $(\mathfrak{R}', \rho')$ *equivalent* if there exists a 1-isomorphism $\iota : \mathfrak{R} \to \mathfrak{R}'$ such that $\rho' \circ \iota$ and ρ are 2-isomorphic 1-morphisms $\mathfrak{R} \to \mathfrak{F}$. Write $[(\mathfrak{R}, \rho)]$ for the equivalence class of $(\mathfrak{R}, \rho)$. If $(\mathfrak{R}, \rho)$ is such a pair and $\mathfrak{S}$ is a closed $\mathbb{K}$-substack of $\mathfrak{R}$ then $(\mathfrak{S}, \rho|_{\mathfrak{S}})$, $(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R}\setminus\mathfrak{S}})$ are pairs of the same kind. Define $\mathrm{SF}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by equivalence classes $[(\mathfrak{R}, \rho)]$ as above, with for each closed $\mathbb{K}$-substack $\mathfrak{S}$ of $\mathfrak{R}$ a relation

$$[(\mathfrak{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R}\setminus\mathfrak{S}})].$$

Define $\underline{\mathrm{SF}}(\mathfrak{F})$ in the same way, but without requiring the 1-morphisms ρ to be representable. Then $\mathrm{SF}(\mathfrak{F}) \subseteq \underline{\mathrm{SF}}(\mathfrak{F})$.

In [28, Def. 3.2] we relate $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{SF}(\mathfrak{F})$.

Definition 2.6. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $C \subseteq \mathfrak{F}(\mathbb{K})$ be constructible. Then $C = \coprod_{i=1}^n \mathfrak{R}_i(\mathbb{K})$, for $\mathfrak{R}_1, \ldots, \mathfrak{R}_n$ finite type $\mathbb{K}$-substacks of $\mathfrak{F}$. Let $\rho_i : \mathfrak{R}_i \to \mathfrak{F}$ be the inclusion 1-morphism. Then $[(\mathfrak{R}_i, \rho_i)] \in \mathrm{SF}(\mathfrak{F})$. Define $\bar\delta_C = \sum_{i=1}^n [(\mathfrak{R}_i, \rho_i)] \in \mathrm{SF}(\mathfrak{F})$. We think of this stack function as the analogue of the characteristic function $\delta_C \in \mathrm{CF}(\mathfrak{F})$ of C. Define a $\mathbb{Q}$-linear map $\iota_{\mathfrak{F}} : \mathrm{CF}(\mathfrak{F}) \to \mathrm{SF}(\mathfrak{F})$ by $\iota_{\mathfrak{F}}(f) = \sum_{0\neq c\in f(\mathfrak{F}(\mathbb{K}))} c \cdot \bar\delta_{f^{-1}(c)}$. For $\mathbb{K}$ of characteristic zero, define a $\mathbb{Q}$-linear map $\pi^{\mathrm{stk}}_{\mathfrak{F}} : \mathrm{SF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{F})$ by

$$\pi^{\mathrm{stk}}_{\mathfrak{F}}\bigl(\textstyle\sum_{i=1}^n c_i[(\mathfrak{R}_i, \rho_i)]\bigr) = \textstyle\sum_{i=1}^n c_i \, \mathrm{CF}^{\mathrm{stk}}(\rho_i) 1_{\mathfrak{R}_i},$$

where $1_{\mathfrak{R}_i}$ is the function 1 in $\mathrm{CF}(\mathfrak{R}_i)$. Then [28, Prop. 3.3] shows $\pi^{\mathrm{stk}}_{\mathfrak{F}} \circ \iota_{\mathfrak{F}}$ is the identity on $\mathrm{CF}(\mathfrak{F})$. Thus, $\iota_{\mathfrak{F}}$ is injective and $\pi^{\mathrm{stk}}_{\mathfrak{F}}$ is surjective. In general $\iota_{\mathfrak{F}}$ is far from being surjective, and $\mathrm{SF}(\mathfrak{F})$ is much larger than $\mathrm{CF}(\mathfrak{F})$.

All the operations of constructible functions in §2.1 extend to stack functions.

Definition 2.7. Define *multiplication* '$\cdot$' on $\underline{\mathrm{SF}}(\mathfrak{F})$ by

$$[(\mathfrak{R}, \rho)] \cdot [(\mathfrak{S}, \sigma)] = [(\mathfrak{R} \times_{\rho,\mathfrak{F},\sigma} \mathfrak{S}, \rho \circ \pi_{\mathfrak{R}})].$$

This extends to a $\mathbb{Q}$-bilinear product $\underline{\mathrm{SF}}(\mathfrak{F}) \times \underline{\mathrm{SF}}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\mathfrak{F})$ which is commutative and associative, and $\mathrm{SF}(\mathfrak{F})$ is closed under '$\cdot$'. Let $\phi : \mathfrak{F} \to \mathfrak{G}$ be a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. Define the *pushforward* $\phi_* : \underline{\mathrm{SF}}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\mathfrak{G})$ by

$$\phi_* : \textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i, \rho_i)] \longmapsto \textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i, \phi \circ \rho_i)].$$

If ϕ is representable then ϕ_* maps $\mathrm{SF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{G})$. For ϕ of finite type, define *pullbacks* $\phi^*:\mathrm{SF}(\mathfrak{G})\to\mathrm{SF}(\mathfrak{F})$, $\phi^*:\underline{\mathrm{SF}}(\mathfrak{G})\to\underline{\mathrm{SF}}(\mathfrak{F})$ by

$$\phi^*:\textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i,\rho_i)]\longmapsto\textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i\times_{\rho_i,\mathfrak{G},\phi}\mathfrak{F},\pi_{\mathfrak{F}})].$$

The *tensor product* $\otimes:\mathrm{SF}(\mathfrak{F})\times\mathrm{SF}(\mathfrak{G})\to\mathrm{SF}(\mathfrak{F}\times\mathfrak{G})$ or $\underline{\mathrm{SF}}(\mathfrak{F})\times\underline{\mathrm{SF}}(\mathfrak{G})\to\underline{\mathrm{SF}}(\mathfrak{F}\times\mathfrak{G})$ is

$$\bigl(\textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i,\rho_i)]\bigr)\otimes\bigl(\textstyle\sum_{j=1}^n d_j[(\mathfrak{S}_j,\sigma_j)]\bigr)=\textstyle\sum_{i,j}c_id_j[(\mathfrak{R}_i\times\mathfrak{S}_j,\rho_i\times\sigma_j)].$$

Here [28, Th. 3.5] is the analogue of Theorem 2.4.

Theorem 2.8. *Let $\mathfrak{E},\mathfrak{F},\mathfrak{G},\mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\beta:\mathfrak{F}\to\mathfrak{G}$, $\gamma:\mathfrak{G}\to\mathfrak{H}$ be 1-morphisms. Then*

$$\begin{aligned}(\gamma\circ\beta)_*&=\gamma_*\circ\beta_*:\underline{\mathrm{SF}}(\mathfrak{F})\to\underline{\mathrm{SF}}(\mathfrak{H}),\quad&(\gamma\circ\beta)_*&=\gamma_*\circ\beta_*:\mathrm{SF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{H}),\\(\gamma\circ\beta)^*&=\beta^*\circ\gamma^*:\underline{\mathrm{SF}}(\mathfrak{H})\to\underline{\mathrm{SF}}(\mathfrak{F}),\quad&(\gamma\circ\beta)^*&=\beta^*\circ\gamma^*:\mathrm{SF}(\mathfrak{H})\to\mathrm{SF}(\mathfrak{F}),\end{aligned}\tag{5}$$

for β,γ representable in the second equation, and of finite type in the third and fourth. If $f,g\in\underline{\mathrm{SF}}(\mathfrak{G})$ and β is finite type then $\beta^*(f\cdot g)=\beta^*(f)\cdot\beta^*(g)$. If

$$\begin{tikzcd}\mathfrak{E}\arrow[r,"\eta"]\arrow[d,"\theta"]&\mathfrak{G}\arrow[d,"\psi"]\\\mathfrak{F}\arrow[r,"\phi"]&\mathfrak{H}\end{tikzcd}$$

is a Cartesian square with η,ϕ representable and θ,ψ of finite type, then the following commutes:

$$\begin{tikzcd}\mathrm{SF}(\mathfrak{E})\arrow[r,"\eta_*"]&\mathrm{SF}(\mathfrak{G})\\\mathrm{SF}(\mathfrak{F})\arrow[r,"\phi_*"]\arrow[u,"\theta^*"]&\mathrm{SF}(\mathfrak{H}).\arrow[u,"\psi^*"]\end{tikzcd}\tag{6}$$

The same applies for $\underline{\mathrm{SF}}(\mathfrak{E}),\ldots,\underline{\mathrm{SF}}(\mathfrak{H})$, without requiring η,ϕ representable.

In [28, Prop. 3.7 & Th. 3.8] we relate pushforwards and pullbacks of stack and constructible functions using $\iota_{\mathfrak{F}},\pi^{\mathrm{stk}}_{\mathfrak{F}}$.

Theorem 2.9. *Let $\mathbb{K}$ have characteristic zero, $\mathfrak{F},\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi:\mathfrak{F}\to\mathfrak{G}$ be a 1-morphism. Then*

(a) $\phi^*\circ\iota_{\mathfrak{G}}=\iota_{\mathfrak{F}}\circ\phi^*:\mathrm{CF}(\mathfrak{G})\to\mathrm{SF}(\mathfrak{F})$ *if ϕ is of finite type;*

(b) $\pi^{\mathrm{stk}}_{\mathfrak{G}}\circ\phi_*=\mathrm{CF}^{\mathrm{stk}}(\phi)\circ\pi^{\mathrm{stk}}_{\mathfrak{F}}:\mathrm{SF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{G})$ *if ϕ is representable; and*

(c) $\pi^{\mathrm{stk}}_{\mathfrak{F}}\circ\phi^*=\phi^*\circ\pi^{\mathrm{stk}}_{\mathfrak{G}}:\mathrm{SF}(\mathfrak{G})\to\mathrm{CF}(\mathfrak{F})$ *if ϕ is of finite type.*

In [28, §5.2] we define *projections* $\Pi^{\mathrm{vi}}_n:\mathrm{SF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{F})$ and $\underline{\mathrm{SF}}(\mathfrak{F})\to\underline{\mathrm{SF}}(\mathfrak{F})$ which project to stack functions whose stabilizer groups have 'virtual rank' n.

In [28, §3] we define *local stack functions* $\mathrm{LSF},\underline{\mathrm{LSF}}(\mathfrak{F})$, the analogue of locally constructible functions. Analogues of Definitions 2.6–2.7 and Theorems 2.8–2.9 hold for $\mathrm{LSF},\underline{\mathrm{LSF}}(\mathfrak{F})$, with differences in which 1-morphisms are required to be of finite type.

2.3 Motivic invariants of Artin stacks

In [28, §4] we extend *motivic* invariants of quasiprojective $\mathbb{K}$-varieties to Artin stacks. We need the following data, [28, Assumptions 4.1 & 6.1].

Assumption 2.10. Suppose Λ is a commutative $\mathbb{Q}$-algebra with identity 1, and

$$\Upsilon : \{\text{isomorphism classes } [X] \text{ of quasiprojective } \mathbb{K}\text{-varieties } X\} \longrightarrow \Lambda$$

a map for $\mathbb{K}$ an algebraically closed field, satisfying the following conditions:

(i) If $Y \subseteq X$ is a closed subvariety then $\Upsilon([X]) = \Upsilon([X \setminus Y]) + \Upsilon([Y])$;

(ii) If X, Y are quasiprojective $\mathbb{K}$-varieties then $\Upsilon([X \times Y]) = \Upsilon([X])\Upsilon([Y])$;

(iii) Write $\ell = \Upsilon([\mathbb{K}])$ in Λ, regarding $\mathbb{K}$ as a $\mathbb{K}$-variety, the affine line (not the point $\operatorname{Spec}\mathbb{K}$). Then ℓ and $\ell^k - 1$ for $k = 1, 2, \ldots$ are invertible in Λ.

Suppose Λ° is a $\mathbb{Q}$-subalgebra of Λ containing the image of Υ and the elements ℓ^{-1} and $(\ell^k + \ell^{k-1} + \cdots + 1)^{-1}$ for $k = 1, 2, \ldots$, but *not* containing $(\ell - 1)^{-1}$. Let Ω be a commutative $\mathbb{Q}$-algebra, and $\pi : \Lambda^\circ \to \Omega$ a surjective $\mathbb{Q}$-algebra morphism, such that $\pi(\ell) = 1$. Define

$$\Theta : \{\text{isomorphism classes } [X] \text{ of quasiprojective } \mathbb{K}\text{-varieties } X\} \longrightarrow \Omega$$

by $\Theta = \pi \circ \Upsilon$. Then $\Theta([\mathbb{K}]) = 1$.

We chose the notation 'ℓ' as in motivic integration $[\mathbb{K}]$ is called the *Tate motive* and written $\mathbb{L}$. We have $\Upsilon\bigl([\mathrm{GL}(m,\mathbb{K})]\bigr) = \ell^{m(m-1)/2}\prod_{k=1}^{m}(\ell^k - 1)$, so (iii) ensures $\Upsilon([\mathrm{GL}(m,\mathbb{K})])$ is invertible in Λ for all $m \geqslant 1$. Here [28, Ex.s 4.3 & 6.3] is an example of suitable $\Lambda, \Upsilon, \ldots$; more are given in [28, §4.1 & §6.1].

Example 2.11. Let $\mathbb{K}$ be an algebraically closed field. Define $\Lambda = \mathbb{Q}(z)$, the algebra of rational functions in z with coefficients in $\mathbb{Q}$. For any quasiprojective $\mathbb{K}$-variety X, let $\Upsilon([X]) = P(X; z)$ be the *virtual Poincaré polynomial* of X. This has a complicated definition in [28, Ex. 4.3] which we do not repeat, involving Deligne's weight filtration when $\operatorname{char}\mathbb{K} = 0$ and the action of the Frobenius on l-adic cohomology when $\operatorname{char}\mathbb{K} > 0$. If X is smooth and projective then $P(X; z)$ is the ordinary Poincaré polynomial $\sum_{k=0}^{2\dim X} b^k(X)z^k$, where $b^k(X)$ is the k^{th} Betti number in l-adic cohomology, for l coprime to $\operatorname{char}\mathbb{K}$. Also $\ell = P(\mathbb{K}; z) = z^2$.

Let Λ° be the subalgebra of $P(z)/Q(z)$ in Λ for which $z \pm 1$ do not divide $Q(z)$. Here are two possibilities for Ω, π. Assumption 2.10 holds in each case.

(a) Set $\Omega = \mathbb{Q}$ and $\pi : f(z) \mapsto f(-1)$. Then $\Theta([X]) = \pi \circ \Upsilon([X])$ is the *Euler characteristic* of X.

(b) Set $\Omega = \mathbb{Q}$ and $\pi : f(z) \mapsto f(1)$. Then $\Theta([X]) = \pi \circ \Upsilon([X])$ is the *sum of the virtual Betti numbers* of X.

We need a few facts about *algebraic* $\mathbb{K}$*-groups*. A good reference is Borel [8]. Following Borel, we define a $\mathbb{K}$-*variety* to be a $\mathbb{K}$-scheme which is reduced, separated, and of finite type, but *not* necessarily irreducible. An algebraic $\mathbb{K}$-group is then a $\mathbb{K}$-variety G with identity $1 \in G$, multiplication $\mu : G \times G \to G$ and inverse $i : G \to G$ (as morphisms of $\mathbb{K}$-varieties) satisfying the usual group axioms. We call G *affine* if it is an affine $\mathbb{K}$-variety. *Special* $\mathbb{K}$-groups are studied by Serre and Grothendieck in [12, §1, §5].

Definition 2.12. An algebraic $\mathbb{K}$-group G is called *special* if every principal G-bundle is locally trivial. Properties of special $\mathbb{K}$-groups can be found in [12, §§1.4, 1.5 & 5.5] and [28, §2.1]. In [28, Lem. 4.6] we show that if Assumption 2.10 holds and G is special then $\Upsilon([G])$ is invertible in Λ.

In [28, Th. 4.9] we extend Υ to Artin stacks, using Definition 2.12.

Theorem 2.13. *Let Assumption 2.10 hold. Then there exists a unique morphism of* $\mathbb{Q}$*-algebras* $\Upsilon' : \underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K}) \to \Lambda$ *such that if* G *is a special algebraic* $\mathbb{K}$*-group acting on a quasiprojective* $\mathbb{K}$*-variety* X *then* $\Upsilon'\bigl([[X/G]]\bigr) = \Upsilon([X])/\Upsilon([G])$.

Thus, if $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers the theorem defines $\Upsilon'([\mathfrak{R}]) \in \Lambda$. Taking Λ, Υ as in Example 2.11 yields the *virtual Poincaré function* $P(\mathfrak{R}; z) = \Upsilon'([\mathfrak{R}])$ of $\mathfrak{R}$, a natural extension of virtual Poincaré polynomials to stacks. In [28, §6] we overcome the restriction that $\Upsilon([G])^{-1}$ exists for all special $\mathbb{K}$-groups G by defining a finer extension of Υ to stacks that keeps track of maximal tori of stabilizer groups, and allows $\Upsilon = \chi$. This can then be used with Θ, Ω in Assumption 2.10.

In [28, §4–§6] we define other classes of stack functions $\underline{\mathrm{SF}}, \underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$, $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda^\circ), \underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Theta, \Omega)$ 'twisted' by the motivic invariants Υ, Θ of Assumption 2.10; the basic facts are explained in [30, §2.5]. All the material of §2.2 applies to these spaces, except that $\pi^{\mathrm{stk}}_{\mathfrak{F}}, \Pi^{\mathrm{vi}}_n$ are not always defined. For the purposes of this paper the differences between these spaces are unimportant, so we shall not explain them.

2.4 Essential stack functions and convergent sums

Motivated by ideas in Behrend and Dhillon [2], we extend the theory of §2.2–§2.3 to include certain local stack functions, and convergent infinite series. This is new material, not contained in [28]. It will be applied in §6.3. We develop it only for $\mathrm{SF}, \mathrm{LSF}(\mathfrak{F})$, but the extensions to $\underline{\mathrm{SF}}, \underline{\mathrm{LSF}}(\mathfrak{F})$ are obvious.

Definition 2.14. Let $\mathbb{K}$ be algebraically closed and $\mathfrak{F}$ an algebraic $\mathbb{K}$-stack with affine stabilizers. As in [35, p. 98-9] a $\mathbb{K}$-stack $\mathfrak{R}$ has a *dimension* $\dim\mathfrak{R}$ in $\mathbb{Z}\cup\{-\infty,\infty\}$, with $\dim[X/G] = \dim X - \dim G$ for a global quotient. For $m \in \mathbb{Z}$ define $\mathrm{SF}, \mathrm{LSF}(\mathfrak{F})_m$ to be the subspaces of $\mathrm{SF}, \mathrm{LSF}(\mathfrak{F})$ spanned by $[(\mathfrak{R}, \rho)]$ with $\dim\mathfrak{R} \leqslant m$, where for $\mathrm{LSF}(\mathfrak{F})_m$ we allow infinite sums $\sum_{i\in I} c_i[(\mathfrak{R}_i, \rho_i)]$ with $\dim\mathfrak{R}_i \leqslant m$ for all $i \in I$. Then $\mathrm{SF}(\mathfrak{F})_m \subseteq \mathrm{SF}(\mathfrak{F})_n$ if $m \leqslant n$, and

$$\mathrm{SF}(\mathfrak{F}) = \bigcup_{m\in\mathbb{Z}} \mathrm{SF}(\mathfrak{F})_m, \; \bigcap_{m\in\mathbb{Z}} \mathrm{LSF}(\mathfrak{F})_m = \{0\}, \; \mathrm{SF}(\mathfrak{F})_m = \mathrm{SF}(\mathfrak{F}) \cap \mathrm{LSF}(\mathfrak{F})_m. \quad (7)$$

Behrend and Dhillon [2, Def. 2.2] define an algebraic $\mathbb{K}$-stack $\mathfrak{R}$ to be *essentially of finite type* if $\mathfrak{R} = \coprod_{n\geqslant 1}\mathfrak{R}_n$ for finite type $\mathbb{K}$-substacks $\mathfrak{R}_n$ with $\dim\mathfrak{R}_n \to -\infty$ as $n\to\infty$. This motivates our next definition, as $[(\mathfrak{R},\rho)]\in \mathrm{LSF}(\mathfrak{F})$ lies in $\mathrm{ESF}(\mathfrak{F})$ if and only if $\mathfrak{R}$ is essentially of finite type.

Definition 2.15. For $\mathfrak{F}$ as above define $\mathrm{ESF}(\mathfrak{F})$ to be the subspace of $f\in \mathrm{LSF}(\mathfrak{F})$ such that for each $m\in\mathbb{Z}$ we may write $f=g+h$ for $g\in\mathrm{SF}(\mathfrak{F})$ and $h\in\mathrm{LSF}(\mathfrak{F})_m$. Then $\mathrm{SF}(\mathfrak{F})\subseteq\mathrm{ESF}(\mathfrak{F})\subseteq\mathrm{LSF}(\mathfrak{F})$. Elements of $\mathrm{ESF}(\mathfrak{F})$ will be called *essential stack functions*. Write $\mathrm{ESF}(\mathfrak{F})_m=\mathrm{ESF}(\mathfrak{F})\cap\mathrm{LSF}(\mathfrak{F})_m$.

Here are two notions of convergence of infinite sums in $\mathrm{ESF},\mathrm{LSF}(\mathfrak{F})$.

Definition 2.16. Let $\mathfrak{F}$ be as above. A possibly infinite sum $\sum_{i\in I} f_i$ with $f_i\in\mathrm{LSF}(\mathfrak{F})$ is called *convergent* if for all finite type $\mathbb{K}$-substacks $\mathfrak{G}$ in $\mathfrak{F}$, the restriction of f_i to $\mathfrak{G}$ is nonzero for only finitely many $i\in I$. Write $f_i=\sum_{a\in A_i} c_i^a[(\mathfrak{R}_i^a,\rho_i^a)]$ for each $i\in I$, where $[(\mathfrak{R}_i^a,\rho_i^a)]$ is supported on the support of f_i for each $a\in A_i$. One can then show $f=\sum_{i\in I}\sum_{a\in A_i} c_i^a[(\mathfrak{R}_i^a,\rho_i^a)]$ is a well-defined element of $\mathrm{LSF}(\mathfrak{F})$. We call f the *limit* of $\sum_{i\in I} f_i$, and write $\sum_{i\in I} f_i=f$. Note that $\mathrm{LSF}(\mathfrak{F})_m$ for $m\in\mathbb{Z}$ are *closed under limits*. The same notion of convergence and limits also works for infinite sums in $\mathrm{LCF}(\mathfrak{F})$.

A sum $\sum_{i\in I} f_i$ with $f_i\in\mathrm{ESF}(\mathfrak{F})$ is called *strongly convergent* if it is convergent, and for all $m\in\mathbb{Z}$ we have $f_i\in\mathrm{ESF}(\mathfrak{F})_m$ for all but finitely many $i\in I$. Write $f=\sum_{i\in I} f_i$ as above, and let $m\in\mathbb{Z}$. Then $f_i\in\mathrm{ESF}(\mathfrak{F})_m$ for all $i\in I\setminus J$, for finite $J\subseteq I$. As $f_j\in\mathrm{ESF}(\mathfrak{F})$ for $j\in J$ we have $f_j=g_j+h_j$ with $g_j\in\mathrm{SF}(\mathfrak{F})$ and $h_j\in\mathrm{LSF}(\mathfrak{F})_m$. Define $g=\sum_{j\in J} g_j$ and $h=\sum_{j\in J} h_j+\sum_{i\in I\setminus J} f_i$. Then g lies in $\mathrm{SF}(\mathfrak{F})$ as g_j does and J is finite, and h lies in $\mathrm{LSF}(\mathfrak{F})_m$ as h_j for $j\in J$ and f_i for $i\in I\setminus J$ do and $\mathrm{LSF}(\mathfrak{F})_m$ is closed under limits. Therefore $f\in\mathrm{ESF}(\mathfrak{F})$, and $\mathrm{ESF}(\mathfrak{F})$ is *closed under strongly convergent limits*.

We modify the first part of Assumption 2.10.

Assumption 2.17. Let Λ be a commutative $\mathbb{Q}$-algebra and $\Lambda_m\subset\Lambda$ for $m\in\mathbb{Z}$ a vector subspace, such that $\Lambda_m\subseteq\Lambda_n$ when $m\leqslant n$ and $\Lambda_m\cdot\Lambda_n\subseteq\Lambda_{m+n}$ for all m,n, with $1\in\Lambda_0$, and $\Lambda=\bigcup_{m\in\mathbb{Z}}\Lambda_m$, $\bigcap_{m\in\mathbb{Z}}\Lambda_m=\{0\}$. Suppose

$$\Upsilon:\{\text{isomorphism classes } [X] \text{ of quasiprojective } \mathbb{K}\text{-varieties } X\}\longrightarrow\Lambda$$

is a map for $\mathbb{K}$ an algebraically closed field, with $\Upsilon([X])\in\Lambda_{\dim X}$, satisfying:

(i) If $Y\subseteq X$ is a closed subvariety then $\Upsilon([X])=\Upsilon([X\setminus Y])+\Upsilon([Y])$;

(ii) If X,Y are quasiprojective $\mathbb{K}$-varieties then $\Upsilon([X\times Y])=\Upsilon([X])\Upsilon([Y])$;

(iii) Write $\ell=\Upsilon([\mathbb{K}])$ in Λ. Then ℓ is invertible in Λ, with $\ell^{-1}\in\Lambda_{-1}$.

(iv) Suppose we are given elements $\lambda_m\in\Lambda/\Lambda_m$ for $m\in\mathbb{Z}$, such that $\lambda_m+\Lambda_n=\lambda_n$ whenever $m<n$, using the inclusion $\Lambda_m\subset\Lambda_n$. Then there exists $\lambda\in\Lambda$ with $\lambda+\Lambda_m=\lambda_m$ for all $m\in\mathbb{Z}$. This λ is unique as $\bigcap_{m\in\mathbb{Z}}\Lambda_m=\{0\}$.

Here are two examples of suitable Υ, Λ, the first modifying Example 2.11.

Example 2.18. Let $\mathbb{K}$ be an algebraically closed field. Define Λ to be the $\mathbb{Q}$-algebra of $\mathbb{Q}$-Laurent series of the form $\sum_{k=-\infty}^{n} c_k z^k$ for $c_k \in \mathbb{Q}$ and $n \in \mathbb{Z}$, that is, power series in z^k where $k \in \mathbb{Z}$ is bounded above but not necessarily below. For $m \in \mathbb{Z}$ define Λ_m to be the vector subspace of series $\sum_{k=-\infty}^{2m} c_k z^k$ involving powers of z bounded above by $2m$. For any quasiprojective $\mathbb{K}$-variety X, let $\Upsilon([X]) = P(X; z)$ be the *virtual Poincaré polynomial* of X, as in Example 2.11. Then Assumption 2.17 holds, with $\ell = z^2$.

Example 2.19. Let $\mathbb{K}$ be an algebraically closed field. As in Craw [14, §2.3] and Behrend and Dhillon [2, §2.1] we define the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{K}})$ of the category of $\mathbb{K}$-varieties $\mathrm{Var}_{\mathbb{K}}$. Setting $\ell = [\mathbb{K}] \in K_0(\mathrm{Var}_{\mathbb{K}})$ we form the ring of fractions $K_0(\mathrm{Var}_{\mathbb{K}})[\ell^{-1}]$ by inverting ℓ. For $m \in \mathbb{Z}$ define $K_0(\mathrm{Var}_{\mathbb{K}})[\ell^{-1}]_m$ to be the subspace generated by elements $\ell^{-n}[X]$ for $n \geqslant 0$ and $\mathbb{K}$-varieties X with $\dim X \leqslant m + n$. This defines a *filtration* of $K_0(\mathrm{Var}_{\mathbb{K}})[\ell^{-1}]$. Write $\hat{K}_0(\mathrm{Var}_{\mathbb{K}})$ to be the *completion* of $K_0(\mathrm{Var}_{\mathbb{K}})[\ell^{-1}]$ with respect to this filtration. It is naturally filtered by subspaces $\hat{K}_0(\mathrm{Var}_{\mathbb{K}})_m$ for $m \in \mathbb{Z}$.

Define a $\mathbb{Q}$-algebra $\Lambda^{\mathrm{uni}} = \hat{K}_0(\mathrm{Var}_{\mathbb{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and set $\Lambda^{\mathrm{uni}}_m = \hat{K}_0(\mathrm{Var}_{\mathbb{K}})_m \otimes_{\mathbb{Z}} \mathbb{Q}$ for $m \in \mathbb{Z}$. Define Υ^{uni} to be the natural map taking a $\mathbb{K}$-variety X to its class $[X]$ in $K_0(\mathrm{Var}_{\mathbb{K}})$ projected to Λ. Then Assumption 2.17 holds for $\Upsilon^{\mathrm{uni}}, \Lambda^{\mathrm{uni}}$, and they are *universal* in that any Υ, Λ satisfying Assumption 2.17 factor via a filtered algebra morphism $\Lambda^{\mathrm{uni}} \to \Lambda$. Note that the ring $\hat{K}_0(\mathrm{Var}_{\mathbb{K}})$ is used all the time in the subject of *motivic integration*, as in Craw [14].

There is a natural notion of *convergence* of infinite sums in Λ.

Definition 2.20. Let Assumption 2.17 hold. A possibly infinite sum $\sum_{i\in I} \lambda_i$ with $\lambda_i \in \Lambda$ is called *convergent* if for all $m \in \mathbb{Z}$ we have $\lambda_i \in \Lambda_m$ for all but finitely many $i \in I$. For $m \in \mathbb{Z}$ define $\lambda_m \in \Lambda/\Lambda_m$ by $\lambda_m = \sum_{j\in J_m} \lambda_j + \Lambda_m$, where J_m is the finite subset of $j \in I$ with $\lambda_j \notin \Lambda_m$. Then $\lambda_m + \Lambda_n = \lambda_n$ whenever $m < n$, so Assumption 2.17(iv) gives a unique $\lambda \in \Lambda$ such that $\lambda + \Lambda_m = \lambda_m$ for all $m \in \mathbb{Z}$. We call λ the *limit* of $\sum_{i\in I} \lambda_i$, and write $\sum_{i\in I} \lambda_i = \lambda$.

For $k \geqslant 1$, the usual geometric series proof shows $\sum_{n\geqslant 1} \ell^{-nk}$ converges in Λ and its limit is $(\ell^k - 1)^{-1}$. Therefore Assumption 2.10(i)–(iii) hold, which implies Theorem 2.13. We can also strengthen it: as $\ell^{-1} \in \Lambda_{-1}$ and $(\ell^k - 1)^{-1} \in \Lambda_{-k}$ we deduce from [28, Lem.s 4.5 & 4.6] that $\Upsilon([G])^{-1} \in \Lambda_{-\dim G}$ for all special $\mathbb{K}$-groups G. Thus in $\Upsilon'([[X/G]]) = \Upsilon([X])/\Upsilon([G])$ we have $\Upsilon([X]) \in \Lambda_{\dim X}$ and $\Upsilon([G])^{-1} \in \Lambda_{-\dim G}$, so $\Upsilon'([[X/G]]) \in \Lambda_{\dim X - \dim G} = \Lambda_{\dim [X/G]}$. As any finite type $\mathfrak{R}$ with affine stabilizers is a disjoint union of $[X/G]$ we deduce:

Theorem 2.21. *Let Assumption 2.17 hold. Then there exists a unique morphism of $\mathbb{Q}$-algebras $\Upsilon' : \underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K}) \to \Lambda$ such that if G is a special algebraic $\mathbb{K}$-group acting on a $\mathbb{K}$-variety X then $\Upsilon'\bigl([[X/G]]\bigr) = \Upsilon([X])/\Upsilon([G])$, and if $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers then $\Upsilon'\bigl([\mathfrak{R}]\bigr) \in \Lambda_{\dim \mathfrak{R}}$.*

The next definition was suggested to the author by Behrend and Dhillon's definition [2, §2.2] of the *motive* of an essentially of finite type $\mathbb{K}$-stack.

Definition 2.22. Let Assumption 2.17 hold, Υ' be as in Theorem 2.21, $\mathfrak{F}$ be as above, and $\Pi : \mathfrak{F} \to \operatorname{Spec}\mathbb{K}$ be the projection 1-morphism. Then Π_* maps $\mathrm{SF}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K})$, so $\Upsilon' \circ \Pi_* : \mathrm{SF}(\mathfrak{F}) \to \Lambda$, with $\Upsilon' \circ \Pi_* : [(\mathfrak{R}, \rho)] \mapsto \Upsilon'([\mathfrak{R}])$. Since $\mathrm{SF}(\mathfrak{F})_m$ is spanned by $[(\mathfrak{R}, \rho)]$ with $\dim \mathfrak{R} \leqslant m$, so that $\Upsilon'\bigl([\mathfrak{R}]\bigr) \in \Lambda_m$ by Theorem 2.21, we have $\Upsilon' \circ \Pi_* : \mathrm{SF}(\mathfrak{F})_m \to \Lambda_m$ for $m \in \mathbb{Z}$.

Let $f \in \mathrm{ESF}(\mathfrak{F})$. For $m \in \mathbb{Z}$ write $f = g_m + h_m$ for $g_m \in \mathrm{SF}(\mathfrak{F})$ and $h_m \in \mathrm{LSF}(\mathfrak{F})_m$. Set $\lambda_m = \Upsilon' \circ \Pi_*(g_m) + \Lambda_m$ in Λ/Λ_m. If g'_m, h'_m are alternative choices of g_m, h_m then $g_m + h_m = g'_m + h'_m$, so $g_m - g'_m = h'_m - h_m$, which lies in $\mathrm{SF}(\mathfrak{F}) \cap \mathrm{LSF}(\mathfrak{F})_m = \mathrm{SF}(\mathfrak{F})_m$ by (7). Thus $\Upsilon' \circ \Pi_*(g_m - g'_m) \in \Lambda_m$, so $\Upsilon' \circ \Pi_*(g_m) + \Lambda_m = \Upsilon' \circ \Pi_*(g'_m) + \Lambda_m$, and λ_m is independent of choices.

If $m < n$ then we may define λ_n using g_m, h_m instead of g_n, h_n, giving $\lambda_m + \Lambda_n = \lambda_n$. Thus Assumption 2.17(iv) gives a unique $\lambda \in \Lambda$ with $\lambda + \Lambda_m = \lambda_m$ for all $m \in \mathbb{Z}$. Define $\Pi_\Lambda(f) = \lambda$. This gives a $\mathbb{Q}$-linear map $\Pi_\Lambda : \mathrm{ESF}(\mathfrak{F}) \to \Lambda$. If $f \in \mathrm{SF}(\mathfrak{F})$ we may take $g_m = f$ and $h_m = 0$ for all $m \in \mathbb{Z}$, giving $\lambda_m = \Upsilon' \circ \Pi_*(f) + \Lambda_m$, so $\lambda = \Upsilon' \circ \Pi_*(f)$ by uniqueness. Thus $\Pi_\Lambda = \Upsilon' \circ \Pi_*$ on $\mathrm{SF}(\mathfrak{F})$. It is easy to show Π_Λ maps $\mathrm{ESF}(\mathfrak{F})_m \to \Lambda_m$ for $m \in \mathbb{Z}$.

This Π_Λ commutes with (strongly) convergent limits in $\mathrm{ESF}(\mathfrak{F})$ and Λ.

Proposition 2.23. *Let Assumption 2.17 hold, $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers, and $\sum_{i\in I} f_i$ be a strongly convergent sum in $\mathrm{ESF}(\mathfrak{F})$ with limit f. Then $\sum_{i\in I} \Pi_\Lambda(f_i)$ is convergent in Λ with limit $\Pi_\Lambda(f)$.*

Proof. Let $m \in \mathbb{Z}$. As $\sum_{i\in I} f_i$ is strongly convergent we have $f_i \in \mathrm{ESF}(\mathfrak{F})_m$ for all $i \in I \setminus J$, where $J \subseteq I$ is finite. But Π_Λ maps $\mathrm{ESF}(\mathfrak{F})_m \to \Lambda_m$, so $\Pi_\Lambda(f_i) \in \Lambda_m$ for all $i \in I \setminus J$ with J finite, and thus $\sum_{i\in I} \Pi_\Lambda(f_i)$ converges in Λ. Let the limits be $f = \sum_{i\in I} f_i$ in $\mathrm{ESF}(\mathfrak{F})$ and $\lambda = \sum_{i\in I} \Pi_\Lambda(f_i)$ in Λ. For each $j \in J$ write $f_j = g_j + h_j$ for $g_j \in \mathrm{SF}(\mathfrak{F})$ and $h_j \in \mathrm{LSF}(\mathfrak{F})_m$. Define $g = \sum_{j\in J} g_j$ and $h = \sum_{j\in J} h_j + \sum_{i\in I\setminus J} f_i$. Then $f = g + h$ with $g \in \mathrm{SF}(\mathfrak{F})$ and $h \in \mathrm{LSF}(\mathfrak{F})_m$. Therefore using Definitions 2.16, 2.20 and 2.22 we have

$$\Pi_\Lambda(f) + \Lambda_m = \Pi_\Lambda(g) + \Lambda_m = \textstyle\sum_{j\in J} \Pi_\Lambda(g_j) + \Lambda_m = \sum_{j\in J} \Pi_\Lambda(f_j) + \Lambda_m = \lambda + \Lambda_m,$$

for all $m \in \mathbb{Z}$. As $\bigcap_{m\in\mathbb{Z}} \Lambda_m = \{0\}$ this forces $\Pi_\Lambda(f) = \lambda$. $\square$

3 Background on configurations from [29–31]

We now recall in §3.1–§3.2 the main definitions and results from [29] on $(I, \preceq)$-configurations and their moduli stacks that we will need later, in §3.3 some facts about algebras of constructible and stack functions from [30], and in §3.4–§3.5 some material on (weak) stability conditions from [31].

3.1 Basic definitions

Here is some notation for *finite posets*, taken from [29, Def.s 3.2, 4.1 & 6.1].

Definition 3.1. A *finite partially ordered set* or *finite poset* $(I,\preceq)$ is a finite set I with a partial order I. Define $J\subseteq I$ to be an *f-set* if $i\in I$ and $h,j\in J$ and $h\preceq i\preceq j$ implies $i\in J$. Define $\mathcal{F}_{(I,\preceq)}$ to be the set of f-sets of I. Define $\mathcal{G}_{(I,\preceq)}$ to be the subset of $(J,K)\in\mathcal{F}_{(I,\preceq)}\times\mathcal{F}_{(I,\preceq)}$ such that $J\subseteq K$, and if $j\in J$ and $k\in K$ with $k\preceq j$, then $k\in J$. Define $\mathcal{H}_{(I,\preceq)}$ to be the subset of $(J,K)\in\mathcal{F}_{(I,\preceq)}\times\mathcal{F}_{(I,\preceq)}$ such that $K\subseteq J$, and if $j\in J$ and $k\in K$ with $k\preceq j$, then $j\in K$.

Let I be a finite set and $\preceq,\trianglelefteq$ partial orders on I such that if $i\preceq j$ then $i\trianglelefteq j$ for $i,j\in I$. Then we say that $\trianglelefteq$ *dominates* $\preceq$. A partial order $\trianglelefteq$ on I is called a *total order* if $i\trianglelefteq j$ or $j\trianglelefteq i$ for all $i,j\in I$. Then $(I,\trianglelefteq)$ is canonically isomorphic to $(\{1,\ldots,n\},\leqslant)$ for $n=|I|$.

We define $(I,\preceq)$-*configurations*, [29, Def. 4.1].

Definition 3.2. Let $(I,\preceq)$ be a finite poset, and use the notation of Definition 3.1. Define an $(I,\preceq)$-*configuration* (σ,ι,π) in an abelian category $\mathcal{A}$ to be maps $\sigma:\mathcal{F}_{(I,\preceq)}\rightarrow\mathrm{Obj}(\mathcal{A})$, $\iota:\mathcal{G}_{(I,\preceq)}\rightarrow\mathrm{Mor}(\mathcal{A})$, and $\pi:\mathcal{H}_{(I,\preceq)}\rightarrow\mathrm{Mor}(\mathcal{A})$, where

(i) $\sigma(J)$ is an object in $\mathcal{A}$ for $J\in\mathcal{F}_{(I,\preceq)}$, with $\sigma(\emptyset)=0$.

(ii) $\iota(J,K):\sigma(J)\rightarrow\sigma(K)$ is injective for $(J,K)\in\mathcal{G}_{(I,\preceq)}$, and $\iota(J,J)=\mathrm{id}_{\sigma(J)}$.

(iii) $\pi(J,K):\sigma(J)\rightarrow\sigma(K)$ is surjective for $(J,K)\in\mathcal{H}_{(I,\preceq)}$, and $\pi(J,J)=\mathrm{id}_{\sigma(J)}$.

These should satisfy the conditions:

(A) Let $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and set $L=K\setminus J$. Then the following is exact in $\mathcal{A}$:

$$0 \longrightarrow \sigma(J) \xrightarrow{\iota(J,K)} \sigma(K) \xrightarrow{\pi(K,L)} \sigma(L) \longrightarrow 0. \tag{8}$$

(B) If $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and $(K,L)\in\mathcal{G}_{(I,\preceq)}$ then $\iota(J,L)=\iota(K,L)\circ\iota(J,K)$.

(C) If $(J,K)\in\mathcal{H}_{(I,\preceq)}$ and $(K,L)\in\mathcal{H}_{(I,\preceq)}$ then $\pi(J,L)=\pi(K,L)\circ\pi(J,K)$.

(D) If $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and $(K,L)\in\mathcal{H}_{(I,\preceq)}$ then

$$\pi(K,L)\circ\iota(J,K)=\iota(J\cap L,L)\circ\pi(J,J\cap L).$$

A *morphism* $\alpha:(\sigma,\iota,\pi)\rightarrow(\sigma',\iota',\pi')$ of $(I,\preceq)$-configurations in $\mathcal{A}$ is a collection of morphisms $\alpha(J):\sigma(J)\rightarrow\sigma'(J)$ for each $J\in\mathcal{F}_{(I,\preceq)}$ satisfying

$$\begin{aligned}\alpha(K)\circ\iota(J,K)&=\iota'(J,K)\circ\alpha(J) &&\text{for all } (J,K)\in\mathcal{G}_{(I,\preceq)},\text{ and}\\ \alpha(K)\circ\pi(J,K)&=\pi'(J,K)\circ\alpha(J) &&\text{for all } (J,K)\in\mathcal{H}_{(I,\preceq)}.\end{aligned}$$

It is an *isomorphism* if $\alpha(J)$ is an isomorphism for all $J\in\mathcal{F}_{(I,\preceq)}$.

In [29, Prop. 4.7] we relate the classes $[\sigma(J)]$ in $K_0(\mathcal{A})$.

Proposition 3.3. *Let (σ,ι,π) be an $(I,\preceq)$-configuration in an abelian category $\mathcal{A}$. Then there exists a unique map $\kappa:I\rightarrow K_0(\mathcal{A})$ such that $[\sigma(J)]=\sum_{j\in J}\kappa(j)$ in $K_0(\mathcal{A})$ for all f-sets $J\subseteq I$.*

Here [29, Def.s 5.1, 5.2] are two ways to construct new configurations.

Definition 3.4. Let $(I,\preceq)$ be a finite poset and $J\in\mathcal{F}_{(I,\preceq)}$. Then $(J,\preceq)$ is also a finite poset, and $\mathcal{F}_{(J,\preceq)},\mathcal{G}_{(J,\preceq)},\mathcal{H}_{(J,\preceq)}\subseteq\mathcal{F}_{(I,\preceq)},\mathcal{G}_{(I,\preceq)},\mathcal{H}_{(I,\preceq)}$. Let (σ,ι,π) be an $(I,\preceq)$-configuration in an abelian category $\mathcal{A}$. Define the $(J,\preceq)$-*subconfiguration* (σ',ι',π') of (σ,ι,π) by $\sigma'=\sigma|_{\mathcal{F}_{(J,\preceq)}}$, $\iota'=\iota|_{\mathcal{G}_{(J,\preceq)}}$ and $\pi'=\pi|_{\mathcal{H}_{(J,\preceq)}}$.

Let $(I,\preceq),(K,\trianglelefteq)$ be finite posets, and $\phi:I\rightarrow K$ be surjective with $i\preceq j$ implies $\phi(i)\trianglelefteq\phi(j)$. Using ϕ^{-1} to pull subsets of K back to I maps $\mathcal{F}_{(K,\trianglelefteq)},\mathcal{G}_{(K,\trianglelefteq)},\mathcal{H}_{(K,\trianglelefteq)}\rightarrow\mathcal{F}_{(I,\preceq)},\mathcal{G}_{(I,\preceq)},\mathcal{H}_{(I,\preceq)}$. Let (σ,ι,π) be an $(I,\preceq)$-configuration in an abelian category $\mathcal{A}$. Define the *quotient* $(K,\trianglelefteq)$-*configuration* $(\tilde\sigma,\tilde\iota,\tilde\pi)$ by $\tilde\sigma(A)=\sigma\bigl(\phi^{-1}(A)\bigr)$ for $A\in\mathcal{F}_{(K,\trianglelefteq)}$, $\tilde\iota(A,B)=\iota\bigl(\phi^{-1}(A),\phi^{-1}(B)\bigr)$ for $(A,B)\in\mathcal{G}_{(K,\trianglelefteq)}$, and $\tilde\pi(A,B)=\pi\bigl(\phi^{-1}(A),\phi^{-1}(B)\bigr)$ for $(A,B)\in\mathcal{H}_{(K,\trianglelefteq)}$.

3.2 Moduli stacks of configurations

Here [29, Assumptions 7.1 & 8.1] is the data we require.

Assumption 3.5. Let $\mathbb{K}$ be an algebraically closed field and $\mathcal{A}$ a $\mathbb{K}$-linear noetherian abelian category with $\mathrm{Ext}^i(X,Y)$ finite-dimensional $\mathbb{K}$-vector spaces for all $X,Y\in\mathcal{A}$ and $i\geqslant 0$. Let $K(\mathcal{A})$ be the quotient of the Grothendieck group $K_0(\mathcal{A})$ by some fixed subgroup. Suppose that if $X\in\mathcal{A}$ with $[X]=0$ in $K(\mathcal{A})$ then $X\cong 0$.

To define moduli stacks of objects or configurations in $\mathcal{A}$, we need some *extra data*, to tell us about algebraic families of objects and morphisms in $\mathcal{A}$, parametrized by a base scheme U. We encode this extra data as a *stack in exact categories* $\mathfrak{F}_{\mathcal{A}}$ on the *category of* $\mathbb{K}$-*schemes* $\mathrm{Sch}_{\mathbb{K}}$, made into a *site* with the *étale topology.* The $\mathbb{K},\mathcal{A},K(\mathcal{A}),\mathfrak{F}_{\mathcal{A}}$ must satisfy some complex additional conditions [29, Assumptions 7.1 & 8.1], which we do not give.

In [29, §9–§10] we define the data $\mathcal{A},K(\mathcal{A}),\mathfrak{F}_{\mathcal{A}}$ in some large classes of examples, and prove Assumption 3.5 holds in each case. Note that [29, 30] did not assume $\mathcal{A}$ *noetherian*, but as in [31] we need this to make τ-semistability well-behaved. All the examples of [29, §9–§10] have $\mathcal{A}$ noetherian.

To apply the constructible functions material of §2.1 we need the ground field $\mathbb{K}$ to have *characteristic zero*, but the stack functions of §2.2–§2.4 work for $\mathbb{K}$ of *arbitrary characteristic*. As we develop the two strands in parallel, in this section for brevity we make the convention that $\mathrm{char}\,\mathbb{K}=0$ in the parts dealing with $\mathrm{CF},\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{char}\,\mathbb{K}$ is arbitrary otherwise.

Definition 3.6. We work in the situation of Assumption 3.5. Define

$$C(\mathcal{A})=\bigl\{[X]\in K(\mathcal{A}):X\in\mathcal{A},\ \ X\not\cong 0\bigr\}\subset K(\mathcal{A}).\tag{9}$$

That is, $C(\mathcal{A})$ is the collection of classes in $K(\mathcal{A})$ of *nonzero objects* $X\in\mathcal{A}$. Note that $C(\mathcal{A})$ is *closed under addition*, as $[X\oplus Y]=[X]+[Y]$. Note also that $0\notin C(\mathcal{A})$, as by Assumption 3.5 if $X\not\cong 0$ then $[X]\neq 0$ in $K(\mathcal{A})$.

In [29, 30] we worked mostly with $\bar C(\mathcal{A})=C(\mathcal{A})\cup\{0\}$, the collection of classes in $K(\mathcal{A})$ of all objects $X\in\mathcal{A}$. But here and in [31] we find $C(\mathcal{A})$ more

useful, as stability conditions will be defined only on nonzero objects. We think of $C(\mathcal{A})$ as the 'positive cone' and $\bar{C}(\mathcal{A})$ as the 'closed positive cone' in $K(\mathcal{A})$.

Define a set of $\mathcal{A}$-*data* to be a triple $(I,\preceq,\kappa)$ such that $(I,\preceq)$ is a finite poset and $\kappa : I \rightarrow C(\mathcal{A})$ a map. We *extend* κ *to the set of subsets of* I by defining $\kappa(J)=\sum_{j\in J}\kappa(j)$. Then $\kappa(J)\in C(\mathcal{A})$ for all $\emptyset\neq J\subseteq I$, as $C(\mathcal{A})$ is closed under addition. Define an $(I,\preceq,\kappa)$-*configuration* to be an $(I,\preceq)$-configuration (σ,ι,π) in $\mathcal{A}$ with $[\sigma(\{i\})]=\kappa(i)$ in $K(\mathcal{A})$ for all $i\in I$. Then $[\sigma(J)]=\kappa(J)$ for all $J\in\mathcal{F}_{(I,\preceq)}$, by Proposition 3.3.

In the situation above, we define the following $\mathbb{K}$-stacks [29, Def.s 7.2 & 7.4]:

- The *moduli stacks* $\mathfrak{Obj}_{\mathcal{A}}$ of *objects in* $\mathcal{A}$, and $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}$ of *objects in* $\mathcal{A}$ *with class* α *in* $K(\mathcal{A})$, for each $\alpha\in\bar{C}(\mathcal{A})$. They are algebraic $\mathbb{K}$-stacks. The underlying geometric spaces $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}),\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$ are the sets of isomorphism classes of objects X in $\mathcal{A}$, with $[X]=\alpha$ for $\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$.
- The *moduli stacks* $\mathfrak{M}(I,\preceq)_{\mathcal{A}}$ of $(I,\preceq)$-*configurations* and $\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}$ of $(I,\preceq,\kappa)$-*configurations in* $\mathcal{A}$, for all finite posets $(I,\preceq)$ and $\kappa : I\rightarrow\bar{C}(\mathcal{A})$. They are algebraic $\mathbb{K}$-stacks. Write $\mathcal{M}(I,\preceq)_{\mathcal{A}},\mathcal{M}(I,\preceq,\kappa)_{\mathcal{A}}$ for the underlying geometric spaces $\mathfrak{M}(I,\preceq)_{\mathcal{A}}(\mathbb{K}),\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}(\mathbb{K})$. Then $\mathcal{M}(I,\preceq)_{\mathcal{A}}$, $\mathcal{M}(I,\preceq,\kappa)_{\mathcal{A}}$ are the *sets of isomorphism classes of* $(I,\preceq)$- *and* $(I,\preceq,\kappa)$-*configurations in* $\mathcal{A}$, by [29, Prop. 7.6].

In [29, Def. 7.7 & Prop. 7.8] we define 1-morphisms of $\mathbb{K}$-stacks, as follows:

- For $(I,\preceq)$ a finite poset, $\kappa : I\rightarrow\bar{C}(\mathcal{A})$ and $J\in\mathcal{F}_{(I,\preceq)}$, we define $\boldsymbol{\sigma}(J):\mathfrak{M}(I,\preceq)_{\mathcal{A}}\rightarrow\mathfrak{Obj}_{\mathcal{A}}$ or $\boldsymbol{\sigma}(J):\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}\rightarrow\mathfrak{Obj}^{\kappa(J)}_{\mathcal{A}}$. The induced maps $\boldsymbol{\sigma}(J)_*:\mathcal{M}(I,\preceq)_{\mathcal{A}}\rightarrow\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ or $\mathcal{M}(I,\preceq,\kappa)_{\mathcal{A}}\rightarrow\mathfrak{Obj}^{\kappa(J)}_{\mathcal{A}}(\mathbb{K})$ act by $\boldsymbol{\sigma}(J)_*:[(\sigma,\iota,\pi)]\mapsto[\sigma(J)]$.
- For $(I,\preceq)$ a finite poset, $\kappa : I\rightarrow\bar{C}(\mathcal{A})$ and $J\in\mathcal{F}_{(I,\preceq)}$, we define the $(J,\preceq)$-*subconfiguration* 1-*morphism* $S(I,\preceq,J):\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}\rightarrow\mathfrak{M}(J,\preceq,\kappa|_J)_{\mathcal{A}}$. Then $S(I,\preceq,J)_*$ takes $[(\sigma,\iota,\pi)]\mapsto[(\sigma',\iota',\pi')]$, for (σ,ι,π) an $(I,\preceq,\kappa)$-configuration in $\mathcal{A}$, and (σ',ι',π') its $(J,\preceq)$-subconfiguration.
- Let $(I,\preceq)$, $(K,\trianglelefteq)$ be finite posets, $\kappa : I\rightarrow\bar{C}(\mathcal{A})$, and $\phi : I\rightarrow K$ be surjective with $i\preceq j$ implies $\phi(i)\trianglelefteq\phi(j)$ for $i,j\in I$. Define $\mu : K\rightarrow\bar{C}(\mathcal{A})$ by $\mu(k)=\kappa(\phi^{-1}(k))$. The *quotient* $(K,\trianglelefteq)$-*configuration* 1-*morphism* is $Q(I,\preceq,K,\trianglelefteq,\phi):\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}\rightarrow\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}}$. Then $Q(I,\preceq,K,\trianglelefteq,\phi)_*$ takes $[(\sigma,\iota,\pi)]\mapsto[(\tilde{\sigma},\tilde{\iota},\tilde{\pi})]$, where (σ,ι,π) is an $(I,\preceq,\kappa)$-configuration in $\mathcal{A}$, and $(\tilde{\sigma},\tilde{\iota},\tilde{\pi})$ its quotient $(K,\trianglelefteq)$-configuration.

3.3 Algebras of constructible and stack functions

Next we summarize parts of [30], which define and study associative multiplications $*$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, based on *Ringel–Hall algebras.*

Definition 3.7. Let Assumption 3.5 hold with $\mathbb{K}$ of characteristic zero. Write $\delta_{[0]}\in\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ for the characteristic function of $[0]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. Following [30,

Def. 4.1], using the diagrams of 1-morphisms and pullbacks, pushforwards

$$\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}} \xleftarrow{\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\})} \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} \xrightarrow{\boldsymbol{\sigma}(\{1,2\})} \mathfrak{Obj}_{\mathcal{A}},$$

$$\begin{tikzcd} \mathrm{CF}\bigl(\mathfrak{Obj}_{\mathcal{A}}\bigr)\times\mathrm{CF}\bigl(\mathfrak{Obj}_{\mathcal{A}}\bigr) \arrow[d, "\otimes"'] \arrow[dr, "(\boldsymbol{\sigma}(\{1\}))^*\cdot(\boldsymbol{\sigma}(\{2\}))^*"] & & \\ \mathrm{CF}\bigl(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}\bigr) \arrow[r, "(\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\}))^*"] & \mathrm{CF}\bigl(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\bigr) \arrow[r, "\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}))"] & \mathrm{CF}\bigl(\mathfrak{Obj}_{\mathcal{A}}\bigr), \end{tikzcd}$$

define a bilinear operation $*:\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})\times\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$f*g=\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1,2\}))\bigl[\boldsymbol{\sigma}(\{1\})^*(f)\cdot\boldsymbol{\sigma}(\{2\})^*(g)\bigr]. \tag{10}$$

Then [30, Th. 4.3] shows $*$ is *associative*, and $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ is a $\mathbb{Q}$-*algebra*, with identity $\delta_{[0]}$ and multiplication $*$.

This extends to *locally constructible functions*, [30, §4.2]. Write $\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})$ for the subspace of $f\in\mathrm{LCF}(\mathfrak{Obj}_{\mathcal{A}})$ supported on $\coprod_{\alpha\in S}\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$ for $S\subset\bar{C}(\mathcal{A})$ a finite subset. Then $*$ in (10) is well-defined on $\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})$, and makes $\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})$ into a $\mathbb{Q}$-algebra containing $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ as a subalgebra.

Following [30, Def. 4.8], write $\mathrm{CF}^{\mathrm{ind}},\mathrm{L\dot{C}F}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ for the vector subspaces of f in $\mathrm{CF},\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})$ *supported on indecomposables*, that is, $f\bigl([X]\bigr)\neq 0$ implies $0\not\cong X$ is indecomposable. Define bilinear brackets $[\,,\,]$ on $\mathrm{CF},\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})$ by $[f,g]=f*g-g*f$. Then [30, Th. 4.9] shows $\mathrm{CF}^{\mathrm{ind}},\mathrm{L\dot{C}F}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ are closed under $[\,,\,]$, and so are $\mathbb{Q}$-Lie algebras.

In [30, §5] we extend much of the above to *stack functions*, as in §2.2. Here are a few of the basic definitions and results.

Definition 3.8. Suppose Assumption 3.5 holds. Define $\mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})$ to be the subspace of $f\in\mathrm{LSF}(\mathfrak{Obj}_{\mathcal{A}})$ supported on $\coprod_{\alpha\in S}\mathfrak{Obj}^{\alpha}_{\mathcal{A}}(\mathbb{K})$ for $S\subset\bar{C}(\mathcal{A})$ finite. By analogy with (10), using $\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}$ define [30, Def. 5.1 & §5.4] bilinear operations $*:\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})\times\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ and $*:\mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})\times\mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})$ by

$$f*g=\boldsymbol{\sigma}(\{1,2\})_*\bigl[(\boldsymbol{\sigma}(\{1\})\times\boldsymbol{\sigma}(\{2\}))^*(f\otimes g)\bigr].$$

Write $\bar{\delta}_{[0]}\in\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ for $\bar{\delta}_C$ in Definition 2.6 with $C=\{[0]\}$. Then [30, Th. 5.2] shows $\mathrm{SF},\mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})$ are $\mathbb{Q}$-algebras with associative multiplication $*$ and identity $\bar{\delta}_{[0]}$. When $\mathbb{K}$ has characteristic zero there are $\mathbb{Q}$-algebra morphisms

$$\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}),\;\;\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}}). \tag{11}$$

As in [30, Def. 5.5] write $\mathrm{SF}_{\mathrm{al}},\mathrm{L\dot{S}F}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ for the subspaces of $\mathrm{SF},\mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})$ spanned by $[(\mathfrak{R},\rho)]$ such that for all $r\in\mathfrak{R}(\mathbb{K})$ with $\rho_*(r)=[X]$, the $\mathbb{K}$-subgroup $\rho_*\bigl(\mathrm{Iso}_{\mathbb{K}}(r)\bigr)$ in $\mathrm{Aut}(X)$ is the $\mathbb{K}$-group of invertible elements in a $\mathbb{K}$-subalgebra of $\mathrm{End}(X)$. Then $\iota_{\mathfrak{Obj}_{\mathcal{A}}}$ in Definition 2.6 maps $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{L\dot{S}F}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, and [30, Prop. 5.6] shows $\mathrm{SF}_{\mathrm{al}},\mathrm{L\dot{S}F}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ are closed under $*$ and so are $\mathbb{Q}$-subalgebras.

Definition 3.9. Suppose Assumption 3.5 holds. Following [30, Def. 5.13], define $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}},\mathrm{L\dot{S}F}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ to be the subspaces of $f\in\mathrm{SF}_{\mathrm{al}},\mathrm{L\dot{S}F}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ with $\Pi_1^{\mathrm{vi}}(f)=f$, where Π_1^{vi} is the operator of [28, §5.2], interpreted as projecting to stack functions 'supported on virtual indecomposables'. Write $[f,g]=f*g-g*f$ for $f,g\in\mathrm{SF}_{\mathrm{al}},\mathrm{L\dot{S}F}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$. As $*$ is associative $[\,,\,]$ satisfies the *Jacobi identity*, and makes $\mathrm{SF}_{\mathrm{al}},\mathrm{L\dot{S}F}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ into $\mathbb{Q}$-*Lie algebras*. Then [30, Th. 5.17] shows $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}},\mathrm{L\dot{S}F}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ are closed under $[\,,\,]$, and are *Lie subalgebras*. When $\operatorname{char}\mathbb{K}=0$, (11) restricts to Lie algebra morphisms

$$\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}),\;\pi^{\mathrm{stk}}_{\mathfrak{Obj}_{\mathcal{A}}}:\mathrm{L\dot{S}F}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\mathrm{L\dot{C}F}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}).$$

All this also works for other stack function spaces on $\mathfrak{Obj}_{\mathcal{A}}$, in particular for $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda),\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}},\Upsilon,\Lambda^\circ)$ and $\bar{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega)$, giving algebras $\bar{\mathrm{SF}},\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ and Lie algebras $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$. In [30, §6] under extra conditions on $\mathcal{A}$, we define (*Lie*) *algebra morphisms* from $\mathrm{SF}_{\mathrm{al}},\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}},*,*)$ to explicit algebras $A(\mathcal{A},\Lambda,\chi),\ldots,C(\mathcal{A},\Omega,\chi)$, which will be important in §6.

3.4 (Weak) stability conditions

We now summarize the material of [31, §4], beginning with [31, Def.s 4.1–4.3].

Definition 3.10. Let $\mathcal{A}$ be an abelian category, $K(\mathcal{A})$ be the quotient of $K_0(\mathcal{A})$ by some fixed subgroup, and $C(\mathcal{A})$ as in (9). Suppose $(T,\leqslant)$ is a totally ordered set, and $\tau:C(\mathcal{A})\rightarrow T$ a map. We call $(\tau,T,\leqslant)$ a *stability condition* on $\mathcal{A}$ if whenever $\alpha,\beta,\gamma\in C(\mathcal{A})$ with $\beta=\alpha+\gamma$ then either $\tau(\alpha)<\tau(\beta)<\tau(\gamma)$, or $\tau(\alpha)>\tau(\beta)>\tau(\gamma)$, or $\tau(\alpha)=\tau(\beta)=\tau(\gamma)$. We call $(\tau,T,\leqslant)$ a *weak stability condition* on $\mathcal{A}$ if whenever $\alpha,\beta,\gamma\in C(\mathcal{A})$ with $\beta=\alpha+\gamma$ then either $\tau(\alpha)\leqslant\tau(\beta)\leqslant\tau(\gamma)$, or $\tau(\alpha)\geqslant\tau(\beta)\geqslant\tau(\gamma)$. The alternative $\tau(\alpha)\leqslant\tau(\beta)\leqslant\tau(\gamma)$ or $\tau(\alpha)\geqslant\tau(\beta)\geqslant\tau(\gamma)$ is called the *weak seesaw inequality*.

We use many ordered sets in the paper: finite posets $(I,\preceq),(J,\lesssim),(K,\trianglelefteq)$ for $(I,\preceq)$-configurations, and now total orders $(T,\leqslant)$ for stability conditions. As the number of order symbols is limited, we will always use '$\leqslant$' for the total order, so that $(\tau,T,\leqslant)$, $(\tilde\tau,\tilde T,\leqslant)$ may denote two different stability conditions, with two *different* total orders on $T,\tilde T$ both denoted by '$\leqslant$'.

Definition 3.11. Let $(\tau,T,\leqslant)$ be a weak stability condition on $\mathcal{A},K(\mathcal{A})$ as above. Then we say that a nonzero object X in $\mathcal{A}$ is

(i) τ-*semistable* if for all $S\subset X$ with $S\not\cong 0,X$ we have $\tau([S])\leqslant\tau([X/S])$;

(ii) τ-*stable* if for all $S\subset X$ with $S\not\cong 0,X$ we have $\tau([S])<\tau([X/S])$; and

(iii) τ-*unstable* if it is not τ-semistable.

Definition 3.12. Let $(\tau,T,\leqslant)$ be a weak stability condition on $\mathcal{A},K(\mathcal{A})$. We say $\mathcal{A}$ is τ-*artinian* if there exist no infinite chains of subobjects $\cdots\subset A_2\subset A_1\subset X$ in $\mathcal{A}$ with $A_{n+1}\neq A_n$ and $\tau([A_{n+1}])\geqslant\tau([A_n/A_{n+1}])$ for all n.

Here is [31, Th. 4.4], based on Rudakov [39, Th. 2]. We call $0=A_0\subset\cdots\subset A_n=X$ in Theorem 3.13 the *Harder–Narasimhan filtration* of X.

Theorem 3.13. *Let $(\tau,T,\leqslant)$ be a weak stability condition on an abelian category $\mathcal{A}$. Suppose $\mathcal{A}$ is noetherian and τ-artinian. Then each $X\in\mathcal{A}$ admits a unique filtration $0=A_0\subset\cdots\subset A_n=X$ for $n\geqslant 0$, such that $S_k=A_k/A_{k-1}$ is τ-semistable for $k=1,\ldots,n$, and $\tau([S_1])>\tau([S_2])>\cdots>\tau([S_n])$.*

We define some notation, [31, Def.s 4.6, 8.1, 4.7 & 4.10].

Definition 3.14. Let Assumption 3.5 hold and $(\tau,T,\leqslant)$ be a weak stability condition on $\mathcal{A}$. Then $\mathfrak{Obj}_{\mathcal{A}}^\alpha$ is an algebraic $\mathbb{K}$-stack for $\alpha\in C(\mathcal{A})$, with $\mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{K})$ the set of isomorphism classes of $X\in\mathcal{A}$ with class α in $K(\mathcal{A})$. Define

$$\begin{aligned}\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)&=\bigl\{[X]\in\mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{K}):X\text{ is }\tau\text{-semistable}\bigr\}\subset\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K}),\\ \mathrm{Obj}_{\mathrm{st}}^\alpha(\tau)&=\bigl\{[X]\in\mathfrak{Obj}_{\mathcal{A}}^\alpha(\mathbb{K}):X\text{ is }\tau\text{-stable}\bigr\}.\end{aligned}$$

For $\mathcal{A}$-data $(I,\preceq,\kappa)$, define $\mathcal{M}_{\mathrm{ss}}(I,\preceq,\kappa,\tau)_{\mathcal{A}}=\bigl\{[(\sigma,\iota,\pi)]\in\mathcal{M}(I,\preceq,\kappa)_{\mathcal{A}}:\sigma(\{i\})\text{ is }\tau\text{-semistable for all }i\in I\bigr\}$. Then $\mathrm{Obj}_{\mathrm{ss}}^\alpha,\mathrm{Obj}_{\mathrm{st}}^\alpha(\tau)$ and $\mathcal{M}_{\mathrm{ss}}(I,\preceq,\kappa,\tau)_{\mathcal{A}}$ are open sets in the natural topologies, and so are *locally constructible sets* in the stacks $\mathfrak{Obj}_{\mathcal{A}},\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}$. Write $\delta_{\mathrm{ss}}^\alpha,\delta_{\mathrm{st}}^\alpha(\tau):\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})\rightarrow\{0,1\}$ and $\delta_{\mathrm{ss}}(I,\preceq,\kappa,\tau):\mathcal{M}(I,\preceq,\kappa)_{\mathcal{A}}\rightarrow\{0,1\}$ for their characteristic functions. Define local stack functions $\bar\delta_{\mathrm{ss}}^\alpha(\tau)=\bar\delta_{\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)}$ and $\bar\delta_{\mathrm{ss}}(I,\preceq,\kappa,\tau)=\bar\delta_{\mathcal{M}_{\mathrm{ss}}(I,\preceq,\kappa,\tau)_{\mathcal{A}}}$, using the local generalization [28, Def. 3.10] of Definition 2.6. Then

$$\begin{aligned}\delta_{\mathrm{ss}}^\alpha,\delta_{\mathrm{st}}^\alpha(\tau)&\in\mathrm{L\dot{C}F}(\mathfrak{Obj}_{\mathcal{A}}), &\delta_{\mathrm{ss}}(I,\preceq,\kappa,\tau)&\in\mathrm{LCF}\bigl(\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}\bigr),\\ \bar\delta_{\mathrm{ss}}^\alpha(\tau)&\in\mathrm{L\dot{S}F}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}), &\bar\delta_{\mathrm{ss}}(I,\preceq,\kappa,\tau)&\in\mathrm{LSF}\bigl(\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}\bigr).\end{aligned}\tag{12}$$

Definition 3.15. Let Assumption 3.5 hold and $(\tau,T,\leqslant)$ be a weak stability condition on $\mathcal{A}$. We call $(\tau,T,\leqslant)$ *permissible* if:

(i) $\mathcal{A}$ is τ-artinian, in the sense of Definition 3.12, and

(ii) $\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)$ is a constructible subset in $\mathfrak{Obj}_{\mathcal{A}}^\alpha$ for all $\alpha\in C(\mathcal{A})$.

Examples of (weak) stability conditions on $\mathcal{A}=\text{mod-}\mathbb{K}Q$ and $\mathcal{A}=\mathrm{coh}(P)$ are given in [31, §4.3–§4.4].

Definition 3.16. Let $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$ be weak stability conditions on an abelian category $\mathcal{A}$, with the same $K(\mathcal{A})$. We say $(\tilde\tau,\tilde T,\leqslant)$ *dominates* $(\tau,T,\leqslant)$ if $\tau(\alpha)\leqslant\tau(\beta)$ implies $\tilde\tau(\alpha)\leqslant\tilde\tau(\beta)$ for all $\alpha,\beta\in C(\mathcal{A})$.

If $(\tau,T,\leqslant)$ is permissible then [31, Th. 4.8] implies $\mathcal{M}_{\mathrm{ss}}(I,\preceq,\kappa,\tau)_{\mathcal{A}}$ is constructible. Together with Definition 3.15(ii) this gives, following (12),

$$\begin{aligned}\delta_{\mathrm{ss}}^\alpha,\delta_{\mathrm{st}}^\alpha(\tau)&\in\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}), &\delta_{\mathrm{ss}}(I,\preceq,\kappa,\tau)&\in\mathrm{CF}\bigl(\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}\bigr),\\ \bar\delta_{\mathrm{ss}}^\alpha(\tau)&\in\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}), &\bar\delta_{\mathrm{ss}}(I,\preceq,\kappa,\tau)&\in\mathrm{SF}\bigl(\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}\bigr).\end{aligned}\tag{13}$$

Now [30] also studies other locally constructible sets: $\mathrm{Obj}_{\mathrm{si}}^{\alpha}(\tau)$ of τ-semistable indecomposable objects in class α, and $\mathcal{M}_{\mathrm{si}}, \mathcal{M}_{\mathrm{st}}, \mathcal{M}_{\mathrm{ss}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{si}}^{\mathrm{b}}, \mathcal{M}_{\mathrm{st}}^{\mathrm{b}}(I, \preceq, \kappa, \tau)_{\mathcal{A}}$ of (best) $(I, \preceq, \kappa)$-configurations (σ, ι, π) whose smallest objects $\sigma(\{i\})$ for $i \in I$ are τ-(semi)stable (indecomposable). Write $\delta_{\mathrm{si}}^{\alpha}(\tau)$, $\delta_{\mathrm{si}}, \delta_{\mathrm{st}}, \delta_{\mathrm{ss}}^{\mathrm{b}}, \delta_{\mathrm{si}}^{\mathrm{b}}, \delta_{\mathrm{st}}^{\mathrm{b}}(I, \preceq, \kappa, \tau)$ for their characteristic functions. We also define local stack function versions $\bar{\delta}_{\mathrm{si}}^{\alpha}(\tau)$ and $\bar{\delta}_{\mathrm{si}}, \ldots, \bar{\delta}_{\mathrm{st}}^{\mathrm{b}}(I, \preceq, \kappa, \tau)$. When $(\tau, T, \leqslant)$ is permissible these sets and functions are all constructible, and the local stack functions are stack functions.

In [31, §5–§8] we prove many identities relating these functions under pushforwards. For example, if $(\tau, T, \leqslant)$ is a permissible stability condition then

$$\begin{aligned} \delta_{\mathrm{st}}^{\alpha}(\tau) = \sum_{\substack{\text{iso. classes}\\ \text{of finite sets } I}} \frac{1}{|I|!} \cdot \sum_{\substack{\preceq, \kappa:\ (I, \preceq, \kappa) \text{ is } \mathcal{A}\text{-data,}\\ \kappa(I) = \alpha,\ \tau \circ \kappa \equiv \tau(\alpha)}} & \mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I)) \delta_{\mathrm{ss}}(I, \preceq, \kappa, \tau) \cdot \\ & \Bigl[\sum_{\text{p.o.s } \lesssim \text{ on } I \text{ dominating } \preceq} n(I, \preceq, \lesssim)\, N(I, \lesssim) \Bigr], \end{aligned} \tag{14}$$

with only finitely many nonzero terms in the sum. This follows from [31, Th.s 6.3 & 6.12], with integers $n(I, \preceq, \lesssim), N(I, \lesssim)$ defined in [31, Def.s 6.1 & 6.9].

3.5 Algebras $\mathcal{H}_{\tau}^{\mathrm{to}}, \mathcal{H}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}$ and elements $\epsilon^{\alpha}(\tau), \bar{\epsilon}^{\alpha}(\tau)$

In [31, §7–§8] we study *subalgebras* $\mathcal{H}_{\tau}^{\mathrm{pa}}, \mathcal{H}_{\tau}^{\mathrm{to}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ in $\mathrm{CF}, \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$.

Definition 3.17. Let Assumption 3.5 hold, and $(\tau, T, \leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. As in [31, Def.s 7.1 & 8.4], define $\mathcal{H}_{\tau}^{\mathrm{pa}}, \mathcal{H}_{\tau}^{\mathrm{to}}$ when $\mathrm{char}\,\mathbb{K} = 0$ and $\bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ for all $\mathbb{K}$ by

$$\mathcal{H}_{\tau}^{\mathrm{pa}} = \bigl\langle \mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I)) \delta_{\mathrm{ss}}(I, \preceq, \kappa, \tau) : (I, \preceq, \kappa) \text{ is } \mathcal{A}\text{-data} \bigr\rangle_{\mathbb{Q}} \subseteq \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}), \tag{15}$$

$$\mathcal{H}_{\tau}^{\mathrm{to}} = \bigl\langle \delta_{[0]}, \delta_{\mathrm{ss}}^{\alpha_1}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\alpha_n}(\tau) : \alpha_1, \ldots, \alpha_n \in C(\mathcal{A}) \bigr\rangle_{\mathbb{Q}} \subseteq \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}), \tag{16}$$

$$\bar{\mathcal{H}}_{\tau}^{\mathrm{pa}} = \bigl\langle \boldsymbol{\sigma}(I)_* \bar{\delta}_{\mathrm{ss}}(I, \preceq, \kappa, \tau) : (I, \preceq, \kappa) \text{ is } \mathcal{A}\text{-data} \bigr\rangle_{\mathbb{Q}} \subseteq \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}), \tag{17}$$

$$\bar{\mathcal{H}}_{\tau}^{\mathrm{to}} = \bigl\langle \bar{\delta}_{[0]}, \bar{\delta}_{\mathrm{ss}}^{\alpha_1}(\tau) * \cdots * \bar{\delta}_{\mathrm{ss}}^{\alpha_n}(\tau) : \alpha_1, \ldots, \alpha_n \in C(\mathcal{A}) \bigr\rangle_{\mathbb{Q}} \subseteq \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}). \tag{18}$$

Here $\langle \cdots \rangle_{\mathbb{Q}}$ is the set of all finite $\mathbb{Q}$-linear combinations of the elements '$\cdots$'.

To relate $\mathcal{H}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}$ and $\mathcal{H}_{\tau}^{\mathrm{to}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$, let $(\{1, \ldots, n\}, \leqslant, \kappa)$ be $\mathcal{A}$-data. Then

$$\begin{aligned} \delta_{\mathrm{ss}}^{\kappa(1)}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\kappa(n)}(\tau) &= \mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(\{1, \ldots, n\})) \delta_{\mathrm{ss}}(\{1, \ldots, n\}, \leqslant, \kappa, \tau), \\ \bar{\delta}_{\mathrm{ss}}^{\kappa(1)}(\tau) * \cdots * \bar{\delta}_{\mathrm{ss}}^{\kappa(n)}(\tau) &= \boldsymbol{\sigma}(\{1, \ldots, n\})_* \bar{\delta}_{\mathrm{ss}}(\{1, \ldots, n\}, \leqslant, \kappa, \tau). \end{aligned} \tag{19}$$

Thus $\mathcal{H}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}$ are the spans of $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I)) \delta_{\mathrm{ss}}(I, \preceq, \kappa, \tau), \boldsymbol{\sigma}(I)_* \bar{\delta}_{\mathrm{ss}}(I, \preceq, \kappa, \tau)$ for $\mathcal{A}$-data $(I, \preceq, \kappa)$ with $\preceq$ a *partial order*, and $\mathcal{H}_{\tau}^{\mathrm{to}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ the spans with $\preceq$ a *total order*. Hence $\mathcal{H}_{\tau}^{\mathrm{to}} \subseteq \mathcal{H}_{\tau}^{\mathrm{pa}}$ and $\bar{\mathcal{H}}_{\tau}^{\mathrm{to}} \subseteq \bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}$. By [31, Prop. 7.2 & Th. 8.5], $\mathcal{H}_{\tau}^{\mathrm{pa}}, \mathcal{H}_{\tau}^{\mathrm{to}}$ are subalgebras of $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}, \bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$ subalgebras of $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, and $\pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\mathrm{stk}}$ induces surjective morphisms $\bar{\mathcal{H}}_{\tau}^{\mathrm{pa}} \to \mathcal{H}_{\tau}^{\mathrm{pa}}$ and $\bar{\mathcal{H}}_{\tau}^{\mathrm{to}} \to \mathcal{H}_{\tau}^{\mathrm{to}}$.

Equation (14) implies that $\delta_{\mathrm{st}}^{\alpha}(\tau) \in \mathcal{H}_{\tau}^{\mathrm{pa}}$. Other identities in [31, §5–§8] imply that the five families of functions $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I)) \delta_{\mathrm{si}}, \delta_{\mathrm{st}}, \delta_{\mathrm{ss}}^{\mathrm{b}}, \delta_{\mathrm{si}}^{\mathrm{b}}, \delta_{\mathrm{st}}^{\mathrm{b}}(*, \tau)$ lie in $\mathcal{H}_{\tau}^{\mathrm{pa}}$

and are alternative spanning sets, so that the identities yield *basis change formulae* in $\mathcal{H}_\tau^{\rm pa}$, and similarly for $\bar{\mathcal{H}}_\tau^{\rm pa}$. One moral is that $\mathcal{H}_\tau^{\rm pa},\bar{\mathcal{H}}_\tau^{\rm pa}$ contain information on both τ-stability and τ-semistability, and so are good tools for studying invariants counting τ-stable and τ-semistable objects, whereas the smaller algebras $\mathcal{H}_\tau^{\rm to},\bar{\mathcal{H}}_\tau^{\rm to}$ only really contain information about τ-semistability.

In [31, Def.s 7.6 & 8.1] we define interesting elements $\epsilon^\alpha(\tau),\bar\epsilon^\alpha(\tau)$.

Definition 3.18. Let Assumption 3.5 hold, and $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. For $\alpha\in C(\mathcal{A})$ define $\epsilon^\alpha(\tau)$ in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ for $\mathrm{char}\,\mathbb{K}=0$, and $\bar\epsilon^\alpha(\tau)$ in $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ for all $\mathbb{K}$, by

$$\epsilon^\alpha(\tau)=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\frac{(-1)^{n-1}}{n}\,\delta_{\rm ss}^{\kappa(1)}(\tau)*\delta_{\rm ss}^{\kappa(2)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa(n)}(\tau),\tag{20}$$

$$\bar\epsilon^\alpha(\tau)=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\frac{(-1)^{n-1}}{n}\,\bar\delta_{\rm ss}^{\kappa(1)}(\tau)*\bar\delta_{\rm ss}^{\kappa(2)}(\tau)*\cdots*\bar\delta_{\rm ss}^{\kappa(n)}(\tau).\tag{21}$$

Then $\pi^{\rm stk}_{\mathfrak{Obj}_{\mathcal{A}}}(\bar\epsilon^\alpha(\tau))=\epsilon^\alpha(\tau)$, and [31, Th.s 7.8 & 8.7] show $\epsilon^\alpha(\tau)\in\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$ and $\bar\epsilon^\alpha(\tau)\in\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$. In [31, Th.s 7.7 & 8.2] we *invert* (20) and (21), giving

$$\delta_{\rm ss}^\alpha(\tau)=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\frac{1}{n!}\,\epsilon^{\kappa(1)}(\tau)*\epsilon^{\kappa(2)}(\tau)*\cdots*\epsilon^{\kappa(n)}(\tau),\tag{22}$$

$$\bar\delta_{\rm ss}^\alpha(\tau)=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\frac{1}{n!}\,\bar\epsilon^{\kappa(1)}(\tau)*\bar\epsilon^{\kappa(2)}(\tau)*\cdots*\bar\epsilon^{\kappa(n)}(\tau).\tag{23}$$

There are *only finitely many nonzero terms* in (20)–(23). We have

$$\epsilon^\alpha(\tau)\bigl([X]\bigr)=\begin{cases}1, & X\text{ is }\tau\text{-stable},\\ \text{in }\mathbb{Q}, & X\text{ is strictly }\tau\text{-semistable and indecomposable},\\ 0, & X\text{ is }\tau\text{-unstable or decomposable},\end{cases}\tag{24}$$

so $\epsilon^\alpha(\tau)$ interpolates between $\delta_{\rm ss}^\alpha,\delta_{\rm st}^\alpha(\tau)$. In [31, eq.s (95) & (123)] we show that

$$\mathcal{H}_\tau^{\rm to}=\bigl\langle\delta_{[0]},\epsilon^{\alpha_1}(\tau)*\cdots*\epsilon^{\alpha_n}(\tau):\alpha_1,\ldots,\alpha_n\in C(\mathcal{A})\bigr\rangle_{\mathbb{Q}},\tag{25}$$

$$\bar{\mathcal{H}}_\tau^{\rm to}=\bigl\langle\bar\delta_{[0]},\bar\epsilon^{\alpha_1}(\tau)*\cdots*\bar\epsilon^{\alpha_n}(\tau):\alpha_1,\ldots,\alpha_n\in C(\mathcal{A})\bigr\rangle_{\mathbb{Q}}.\tag{26}$$

Thus $\epsilon^\alpha,\bar\epsilon^\alpha(\tau)$ are *alternative generators* for $\mathcal{H}_\tau^{\rm to},\bar{\mathcal{H}}_\tau^{\rm to}$ in $\mathrm{CF}^{\rm ind},\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$.

In [31, Def.s 7.1 & 8.9] we define *Lie algebras* $\mathcal{L}_\tau^{\rm pa},\mathcal{L}_\tau^{\rm to},\bar{\mathcal{L}}_\tau^{\rm pa},\bar{\mathcal{L}}_\tau^{\rm to}$.

Definition 3.19. Let Assumption 3.5 hold, and $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. Define $\mathcal{L}_\tau^{\rm pa}=\mathcal{H}_\tau^{\rm pa}\cap\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$, $\mathcal{L}_\tau^{\rm to}=\mathcal{H}_\tau^{\rm to}\cap\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$ and $\bar{\mathcal{L}}_\tau^{\rm pa}=\bar{\mathcal{H}}_\tau^{\rm pa}\cap\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$. Then $\mathcal{L}_\tau^{\rm pa},\mathcal{L}_\tau^{\rm to},\bar{\mathcal{L}}_\tau^{\rm pa}$ are *Lie subalgebras* of $\mathrm{CF}^{\rm ind},\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$. We also define $\bar{\mathcal{L}}_\tau^{\rm to}$ to be the Lie subalgebra

of $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\bar{\epsilon}^\alpha(\tau)$ for all $\alpha \in C(\mathcal{A})$. Then $\mathcal{L}_\tau^{\mathrm{to}} \subseteq \mathcal{L}_\tau^{\mathrm{pa}}$, $\bar{\mathcal{L}}_\tau^{\mathrm{to}} \subseteq \bar{\mathcal{L}}_\tau^{\mathrm{pa}}$, and $\pi_{\mathfrak{Obj}_{\mathcal{A}}}^{\mathrm{stk}}$ induces surjective morphisms $\bar{\mathcal{L}}_\tau^{\mathrm{pa}} \to \mathcal{L}_\tau^{\mathrm{pa}}$ and $\bar{\mathcal{L}}_\tau^{\mathrm{to}} \to \mathcal{L}_\tau^{\mathrm{to}}$.

In [31, Prop. 7.5 & Def. 8.9] we show $\mathcal{L}_\tau^{\mathrm{pa}}, \bar{\mathcal{L}}_\tau^{\mathrm{pa}}$ are spanned by elements $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(I))\delta_{\mathrm{si}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)$, $\boldsymbol{\sigma}(I)_*\bar{\delta}_{\mathrm{si}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)$ for connected $(I,\preceq)$, and deduce that $\mathcal{L}_\tau^{\mathrm{pa}}, \bar{\mathcal{L}}_\tau^{\mathrm{pa}}$ generate $\mathcal{H}_\tau^{\mathrm{pa}}, \bar{\mathcal{H}}_\tau^{\mathrm{pa}}$ as algebras. This induces surjective morphisms $\Phi_\tau^{\mathrm{pa}} : U(\mathcal{L}_\tau^{\mathrm{pa}}) \to \mathcal{H}_\tau^{\mathrm{pa}}$, $\bar{\Phi}_\tau^{\mathrm{pa}} : U(\bar{\mathcal{L}}_\tau^{\mathrm{pa}}) \to \bar{\mathcal{H}}_\tau^{\mathrm{pa}}$, where $U(\mathcal{L}_\tau^{\mathrm{pa}}), U(\bar{\mathcal{L}}_\tau^{\mathrm{pa}})$ are the *universal enveloping algebras* of $\mathcal{L}_\tau^{\mathrm{pa}}, \bar{\mathcal{L}}_\tau^{\mathrm{pa}}$. Moreover Φ_τ^{pa} is an isomorphism.

In [31, Cor. 7.9] we show $\mathcal{L}_\tau^{\mathrm{to}}$ is the Lie subalgebra of $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ generated by the $\epsilon^\alpha(\tau)$ for $\alpha \in C(\mathcal{A})$. So from (25)–(26) we see $\mathcal{L}_\tau^{\mathrm{to}}, \bar{\mathcal{L}}_\tau^{\mathrm{to}}$ generate $\mathcal{H}_\tau^{\mathrm{to}}, \bar{\mathcal{H}}_\tau^{\mathrm{to}}$ as algebras, giving surjective algebra morphisms $\Phi_\tau^{\mathrm{to}} : U(\mathcal{L}_\tau^{\mathrm{to}}) \to \mathcal{H}_\tau^{\mathrm{to}}$ and $\bar{\Phi}_\tau^{\mathrm{to}} : U(\bar{\mathcal{L}}_\tau^{\mathrm{to}}) \to \bar{\mathcal{H}}_\tau^{\mathrm{to}}$. Again, Φ_τ^{to} is an isomorphism.

4 Transformation coefficients $S, T, U(*, \tau, \tilde{\tau})$

Let $\mathcal{A}$ satisfy Assumption 3.5 and $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ be permissible weak stability conditions on $\mathcal{A}$. In §5 we shall prove *transformation laws* from $(\tau, T, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$ of the form, for all $\alpha \in C(\mathcal{A})$ and $\mathcal{A}$-data $(K, \trianglelefteq, \mu)$,

$$\sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde{\tau})\cdot \\ &\delta_{\mathrm{ss}}^{\kappa(1)}(\tau) * \delta_{\mathrm{ss}}^{\kappa(2)}(\tau) * \cdots * \delta_{\mathrm{ss}}^{\kappa(n)}(\tau) = \delta_{\mathrm{ss}}^\alpha(\tilde{\tau}),\end{aligned} \tag{27}$$

$$\sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa) \text{ is } \mathcal{A}\text{-data},\\ (I,\preceq,K,\phi) \text{ is dominant},\\ \trianglelefteq = \mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k) \text{ for } k\in K}} \begin{aligned}[t] &T(I,\preceq,\kappa,K,\phi,\tau,\tilde{\tau})\cdot \\ &\mathrm{CF}^{\mathrm{stk}}\bigl(Q(I,\preceq,K,\trianglelefteq,\phi)\bigr) \\ &\delta_{\mathrm{ss}}(I,\preceq,\kappa,\tau) = \delta_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tilde{\tau}),\end{aligned} \tag{28}$$

$$\sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde{\tau})\cdot \\ &\epsilon^{\kappa(1)}(\tau) * \epsilon^{\kappa(2)}(\tau) * \cdots * \epsilon^{\kappa(n)}(\tau) = \epsilon^\alpha(\tilde{\tau}),\end{aligned} \tag{29}$$

and analogues of these equations for stack functions $\bar{\delta}_{\mathrm{ss}}^\alpha(\tilde{\tau}), \bar{\delta}_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tilde{\tau}), \bar{\epsilon}^\alpha(\tilde{\tau})$. Here $S, T, U(*,\tau,\tilde{\tau})$ are explicit *transformation coefficients* in $\mathbb{Q}$, and $*$ is as in §3.3. This section will define and study the $S, T, U(*,\tau,\tilde{\tau})$, in preparation for §5. Its definitions and results are all *combinatorial* in nature.

Suppose Assumption 3.5 holds, and $(\tau, T, \leqslant)$ is a weak stability condition on $\mathcal{A}$. Then from §3 the following hold:

Condition 4.1. (i) $K(\mathcal{A})$ is an abelian group.

(ii) $C(\mathcal{A})$ is a subset of $K(\mathcal{A})$, closed under addition and not containing zero.

(iii) a set of $\mathcal{A}$-*data* $(I,\preceq,\kappa)$ is by definition a finite poset $(I,\preceq)$ and a map $\kappa : I \to C(\mathcal{A})$. For $J \subseteq I$ we define $\kappa(J) = \sum_{j\in J}\kappa(j)$.

(iv) $(T, \leqslant)$ is a total order.

(v) $\tau : C(\mathcal{A}) \to T$ is a map such that if $\alpha, \beta, \gamma \in C(\mathcal{A})$ with $\beta = \alpha + \gamma$ then $\tau(\alpha) \leqslant \tau(\beta) \leqslant \tau(\gamma)$ or $\tau(\alpha) \geqslant \tau(\beta) \geqslant \tau(\gamma)$.

More generally, if we have more than one weak stability condition we shall say that 'Condition 4.1 holds for $(\tilde\tau,\tilde T,\leqslant),(\hat\tau,\hat T,\leqslant),\ldots$' if $(\tilde T,\leqslant),(\hat T,\leqslant),\ldots$ are total orders, and (v) holds for $\tilde\tau: C(\mathcal{A})\to\tilde T$, $\hat\tau: C(\mathcal{A})\to\hat T,\ldots$.

These are the only properties of $\mathcal{A},K(\mathcal{A}),(\tau,T,\leqslant)$ that we will use in this section. We shall not use Assumption 3.5, or suppose $(\tau,T,\leqslant)$ is permissible.

4.1 Basic definitions and main results

We begin by defining *transformation coefficients* $S,T,U(*,\tau,\tilde\tau)$.

Definition 4.2. Suppose Condition 4.1 holds for $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$, and let $(\{1,\ldots,n\},\leqslant,\kappa)$ be $\mathcal{A}$-data. If for all $i=1,\ldots,n-1$ we have either

(a) $\tau\circ\kappa(i)\leqslant\tau\circ\kappa(i+1)$ and $\tilde\tau\circ\kappa(\{1,\ldots,i\})>\tilde\tau\circ\kappa(\{i+1,\ldots,n\})$ or

(b) $\tau\circ\kappa(i)>\tau\circ\kappa(i+1)$ and $\tilde\tau\circ\kappa(\{1,\ldots,i\})\leqslant\tilde\tau\circ\kappa(\{i+1,\ldots,n\})$,

then define $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)=(-1)^r$, where r is the number of $i=1,\ldots,n-1$ satisfying (a). Otherwise define $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)=0$.

If $(I,\preceq,\kappa)$ is $\mathcal{A}$-data with $\preceq$ a *total order*, there is a unique bijection $\phi:\{1,\ldots,n\}\to I$ with $n=|I|$ and $\phi_*(\leqslant)=\preceq$, and $(\{1,\ldots,n\},\leqslant,\kappa\circ\phi)$ is $\mathcal{A}$-data. Define $S(I,\preceq,\kappa,\tau,\tilde\tau)=S(\{1,\ldots,n\},\leqslant,\kappa\circ\phi,\tau,\tilde\tau)$.

Definition 4.3. Let $(I,\preceq)$ be a finite poset, K a finite set, and $\phi:I\to K$ a surjective map. We call $(I,\preceq,K,\phi)$ *dominant* if there exists a partial order $\trianglelefteq$ on K such that $(\phi^{-1}(\{k\}),\preceq)$ is a total order for all $k\in K$, and if $i,j\in I$ with $\phi(i)\ne\phi(j)$ then $i\preceq j$ if and only if $\phi(i)\trianglelefteq\phi(j)$. Then $\trianglelefteq$ is determined uniquely by $(I,\preceq,K,\phi)$, and we write $\trianglelefteq=\mathcal{P}(I,\preceq,K,\phi)$. Note that $i\preceq j$ implies $\phi(i)\trianglelefteq\phi(j)$ for $i,j\in I$. We use the notation 'dominant' as if $\lesssim$ is a partial order on I dominating $\preceq$ with $i\lesssim j$ implies $\phi(i)\trianglelefteq\phi(j)$, then $\lesssim\,=\,\preceq$. That is, the partial order $\preceq$ is as strong as it can be, given $I,K,\phi,\trianglelefteq$.

Now suppose Condition 4.1 holds for $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$, and let $(I,\preceq,\kappa)$ be $\mathcal{A}$-data and $(I,\preceq,K,\phi)$ be dominant. Define

$$T(I,\preceq,\kappa,K,\phi,\tau,\tilde\tau)=\textstyle\prod_{k\in K}S\bigl(\phi^{-1}(\{k\}),\preceq\vert_{\phi^{-1}(\{k\})},\kappa\vert_{\phi^{-1}(\{k\})},\tau,\tilde\tau\bigr).\qquad(30)$$

Definition 4.4. Suppose Condition 4.1 holds for $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$, and let $(\{1,\ldots,n\},\leqslant,\kappa)$ be $\mathcal{A}$-data. Define

$$\begin{gathered}U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)=\sum_{1\leqslant l\leqslant m\leqslant n}\\ \sum_{\substack{\text{surjective }\psi:\{1,\ldots,n\}\to\{1,\ldots,m\}\\ \text{and }\xi:\{1,\ldots,m\}\to\{1,\ldots,l\}:\\ i\leqslant j\text{ implies }\psi(i)\leqslant\psi(j),\ i\leqslant j\text{ implies }\xi(i)\leqslant\xi(j).\\ \text{Define }\lambda:\{1,\ldots,m\}\to C(\mathcal{A})\text{ by }\lambda(b)=\kappa(\psi^{-1}(b)).\\ \text{Define }\mu:\{1,\ldots,l\}\to C(\mathcal{A})\text{ by }\mu(a)=\lambda(\xi^{-1}(a)).\\ \text{Then }\tau\circ\kappa\equiv\tau\circ\lambda\circ\mu:I\to T\text{ and }\tilde\tau\circ\mu\equiv\tilde\tau(\alpha)}}\begin{aligned}[t]&\prod_{a=1}^{l}S(\xi^{-1}(\{a\}),\leqslant,\lambda,\tau,\tilde\tau)\cdot\\ &\frac{(-1)^{l-1}}{l}\cdot\prod_{b=1}^{m}\frac{1}{|\psi^{-1}(b)|!}\,.\end{aligned}\end{gathered}\qquad(31)$$

If $(I,\preceq,\kappa)$ is $\mathcal{A}$-data with $\preceq$ a *total order*, there is a unique bijection $\phi:\{1,\ldots,n\}\rightarrow I$ with $n=|I|$ and $\phi_*(\leqslant)=\preceq$, and $(\{1,\ldots,n\},\leqslant,\kappa\circ\phi)$ is $\mathcal{A}$-data. Define $U(I,\preceq,\kappa,\tau,\tilde\tau)=U(\{1,\ldots,n\},\leqslant,\kappa\circ\phi,\tau,\tilde\tau)$.

Then $S,T(*,\tau,\tilde\tau)$ are 1,0 or -1, and $U(*,\tau,\tilde\tau)$ lies in $\mathbb{Q}$. Here is our main result on the properties of the $S(*,\tau,\tilde\tau)$. It is easy to see (32) and (33) are necessary if (27) is to hold: (32) means (27) reduces to $\delta_{\rm ss}^\alpha(\tau)=\delta_{\rm ss}^\alpha(\tau)$ when $(\tilde\tau,\tilde T,\leqslant)=(\tau,T,\leqslant)$, and (33) is the condition for transforming from $(\tau,T,\leqslant)$ to $(\hat\tau,\hat T,\leqslant)$ and then from $(\hat\tau,\hat T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ using (27) to give the same answer as transforming from $(\tau,T,\leqslant)$ directly to $(\tilde\tau,\tilde T,\leqslant)$.

Theorem 4.5. *Let Condition 4.1 hold for* $(\tau,T,\leqslant),(\hat\tau,\hat T,\leqslant)$ *and* $(\tilde\tau,\tilde T,\leqslant)$. *Suppose* $(\{1,\ldots,n\},\leqslant,\kappa)$ *is* $\mathcal{A}$*-data. Then*

$$S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tau)=\begin{cases}1, & n=1,\\ 0, & \textit{otherwise},\end{cases}\tag{32}$$

$$\sum_{m=1}^{n}\sum_{\substack{\psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}:\\ \psi\textit{ is surjective},\\ 1\leqslant i\leqslant j\leqslant n\textit{ implies }\psi(i)\leqslant\psi(j),\\ \textit{define }\lambda:\{1,\ldots,m\}\rightarrow C(\mathcal{A})\\ \textit{by }\lambda(k)=\kappa(\psi^{-1}(k))}}\begin{aligned}[t]&S(\{1,\ldots,m\},\leqslant,\lambda,\hat\tau,\tilde\tau)\cdot\\ &\prod_{k=1}^{m}S\bigl(\psi^{-1}(\{k\}),\leqslant,\kappa|_{\psi^{-1}(\{k\})},\tau,\hat\tau\bigr)\\ &\quad=S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau).\end{aligned}\tag{33}$$

We also give a criterion for when $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\neq0$ based on the minimum and maximum values of the $\tau\circ\kappa(i)$.

Theorem 4.6. *Suppose Condition 4.1 holds for* $(\tau,T,\leqslant)$ *and* $(\tilde\tau,\tilde T,\leqslant)$*, and* $(\{1,\ldots,n\},\leqslant,\kappa)$ *is* $\mathcal{A}$*-data for* $n\geqslant1$ *with* $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\neq0$. *Then there exist* $k,l=1,\ldots,n$ *such that* $\tau\circ\kappa(k)\leqslant\tau\circ\kappa(i)\leqslant\tau\circ\kappa(l)$ *for all* $i=1,\ldots,n$*, and* $\tilde\tau\circ\kappa(k)\geqslant\tilde\tau\circ\kappa(\{1,\ldots,n\})\geqslant\tilde\tau\circ\kappa(l)$.

Here are the analogues of Theorem 4.5 for the $T,U(*,\tau,\tilde\tau)$. Again, it is easy to see (34)–(35) and (36)–(37) are necessary if (28) and (29) are to hold.

Theorem 4.7. *Let Condition 4.1 hold for* $(\tau,T,\leqslant),(\hat\tau,\hat T,\leqslant)$ *and* $(\tilde\tau,\tilde T,\leqslant)$. *Suppose* $(I,\preceq,\kappa)$ *is* $\mathcal{A}$*-data, and* $\phi:I\rightarrow K$ *is surjective with* $(I,\preceq,K,\phi)$ *dominant. Then*

$$T(I,\preceq,\kappa,K,\phi,\tau,\tau)=\begin{cases}1, & \phi\textit{ is a bijection},\\ 0, & \textit{otherwise},\end{cases}\tag{34}$$

$$\sum_{\substack{\textit{iso.}\\ \textit{classes}\\ \textit{of finite}\\ \textit{sets }J}}\frac{1}{|J|!}\cdot\sum_{\substack{\psi:I\rightarrow J,\ \xi:J\rightarrow K\textit{ surjective: }\phi=\xi\circ\psi,\\ (I,\preceq,J,\psi)\textit{ is dominant, }\lesssim=\mathcal{P}(I,\preceq,J,\psi),\\ \textit{define }\lambda:J\rightarrow K(\mathcal{A})\textit{ by }\lambda(j)=\kappa(\psi^{-1}(j))}}\begin{aligned}[t]&T(I,\preceq,\kappa,J,\psi,\tau,\hat\tau)\cdot\\ &T(J,\lesssim,\lambda,K,\xi,\hat\tau,\tilde\tau)=\\ &\ T(I,\preceq,\kappa,K,\phi,\tau,\tilde\tau).\end{aligned}\tag{35}$$

This follows immediately from Theorem 4.5: to get (34) and (35) we take the product over $k\in K$ of equations (32) and (33) with $(\{1,\ldots,n\},\leqslant,\kappa)$ replaced by $(\phi^{-1}(\{k\}),\preceq|_{\phi^{-1}(\{k\})},\kappa|_{\phi^{-1}(\{k\})})$, and use (30) and some simple combinatorics. We leave the details to the reader.

Theorem 4.8. *Let Condition 4.1 hold for $(\tau,T,\leqslant),(\hat\tau,\hat T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$. Suppose $(\{1,\ldots,n\},\leqslant,\kappa)$ is $\mathcal{A}$-data. Then*

$$U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tau)=\begin{cases}1, & n=1,\\ 0, & \text{otherwise,}\end{cases}\tag{36}$$

$$\sum_{m=1}^{n}\;\sum_{\substack{\psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}:\\ \psi\text{ is surjective,}\\ i\leqslant j\text{ implies }\psi(i)\leqslant\psi(j),\\ \text{define }\lambda:\{1,\ldots,m\}\rightarrow C(\mathcal{A})\\ \text{by }\lambda(k)=\kappa(\psi^{-1}(k))}}\begin{aligned}[t]&U(\{1,\ldots,m\},\leqslant,\lambda,\hat\tau,\tilde\tau)\cdot\\ &\textstyle\prod_{k=1}^{m}U\bigl(\psi^{-1}(\{k\}),\leqslant,\kappa|_{\psi^{-1}(\{k\})},\tau,\hat\tau\bigr)\\ &\quad=U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau).\end{aligned}\tag{37}$$

Theorem 4.8 follows directly from Theorem 4.5 and the following proposition, which is a combinatorial consequence of the proof in [31, Th. 7.7] that (22) is the inverse of (20).

Proposition 4.9. *Let $1\leqslant l\leqslant n$ and $\phi:\{1,\ldots,n\}\rightarrow\{1,\ldots,l\}$ be surjective with $1\leqslant i\leqslant j\leqslant n$ implies $\phi(i)\leqslant\phi(j)$. Then*

$$\sum_{m=l}^{n}\;\sum_{\substack{\psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}\\ \text{and }\xi:\{1,\ldots,m\}\rightarrow\{1,\ldots,l\}\\ \text{surjective with }\phi=\xi\circ\psi:\\ i\leqslant j\text{ implies }\psi(i)\leqslant\psi(j),\\ i\leqslant j\text{ implies }\xi(i)\leqslant\xi(j)}}\prod_{a=1}^{l}\frac{1}{|\xi^{-1}(a)|!}\cdot\prod_{b=1}^{m}\frac{(-1)^{|\psi^{-1}(b)|-1}}{|\psi^{-1}(b)|}=\begin{cases}1, & l=n,\\ 0, & \text{otherwise,}\end{cases}$$

$$\sum_{m=l}^{n}\;\sum_{\substack{\psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}\\ \text{and }\xi:\{1,\ldots,m\}\rightarrow\{1,\ldots,l\}\\ \text{surjective with }\phi=\xi\circ\psi:\\ i\leqslant j\text{ implies }\psi(i)\leqslant\psi(j),\\ i\leqslant j\text{ implies }\xi(i)\leqslant\xi(j)}}\prod_{a=1}^{l}\frac{(-1)^{|\xi^{-1}(a)|-1}}{|\xi^{-1}(a)|}\cdot\prod_{b=1}^{m}\frac{1}{|\psi^{-1}(b)|!}=\begin{cases}1, & l=n,\\ 0, & \text{otherwise.}\end{cases}$$

In the remainder of the section we prove Theorems 4.5 and 4.6.

4.2 Proof of Theorem 4.6 and equation (32)

In the situation of Theorem 4.6, as $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\neq 0$ one of Definition 4.2(a) or (b) holds for each $i=1,\ldots,n-1$. If $n=1$ the result is trivial, so suppose $n>1$. Let $k=1,\ldots,n$ be the unique value such that $\tau\circ\kappa(k)$ is minimal amongst all $\tau\circ\kappa(i)$ in the total order $(T,\leqslant)$, and k is least with this condition. Then $\tau\circ\kappa(k)\leqslant\tau\circ\kappa(i)$ for all $i=1,\ldots,n$. To prove $\tilde\tau\circ\kappa(k)\geqslant\tilde\tau\circ\kappa(\{1,\ldots,n\})$, we divide into three cases (i) $k=1$, (ii) $k=n$, and (iii) $1<k<n$.

In case (i) we have $\tau\circ\kappa(1)\leqslant\tau\circ\kappa(2)$, which excludes (b) for $i=1$. Hence (a) holds, giving $\tilde\tau\circ\kappa(\{1\})>\tilde\tau\circ\kappa(\{2,\ldots,n\})$, so $\tilde\tau\circ\kappa(1)\geqslant\tilde\tau\circ\kappa(\{1,\ldots,n\})$ by Condition 4.1(v), as we want. In (ii) we have $\tau\circ\kappa(n-1)>\tau\circ\kappa(n)$, since $k=n$ is least with the minimal value of $\tau\circ\kappa(i)$. This excludes (a) for $i=n-1$, so (b) holds, giving $\tilde\tau\circ\kappa(\{1,\ldots,n-1\})\leqslant\tilde\tau\circ\kappa(\{n\})$. Condition 4.1(v) then gives $\tilde\tau\circ\kappa(n)\geqslant\tilde\tau\circ\kappa(\{1,\ldots,n\})$, as we want.

Case (iii) implies that $\tau\circ\kappa(k-1)>\tau\circ\kappa(k)\leqslant\tau\circ\kappa(k+1)$, since k is least with the minimal value of $\tau\circ\kappa(i)$. Therefore (b) holds for $i=k-1$ and (a) holds for $i=k$, giving $\tilde\tau\circ\kappa(\{1,\ldots,k-1\})\leqslant\tilde\tau\circ\kappa(\{1,\ldots,n\})<\tilde\tau\circ\kappa(\{1,\ldots,k\})$ by Condition 4.1(v). As $\tilde\tau\circ\kappa(\{1,\ldots,k-1\})<\tilde\tau\circ\kappa(\{1,\ldots,k\})$ Condition 4.1(v) gives $\tilde\tau\circ\kappa(\{1,\ldots,k\})\leqslant\tilde\tau\circ\kappa(k)$, and so $\tilde\tau(k)\geqslant\tilde\tau\circ\kappa(\{1,\ldots,n\})$, as we want. For the second part, let $l=1,\ldots,n$ be the unique value such that $\tau\circ\kappa(l)$ is maximal amongst all $\tau\circ\kappa(i)$ in the total order $(T,\leqslant)$, and l is greatest with this condition, and argue in the same way. This proves Theorem 4.6.

The first line of (32) is immediate from Definition 4.2. For the second, suppose for a contradiction that $n>1$ and $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tau)\neq 0$. Then Theorem 4.6 gives k,l with $\tau\circ\kappa(k)\leqslant\tau\circ\kappa(i)\leqslant\tau\circ\kappa(l)$ for all i and $\tau\circ\kappa(k)\geqslant\tau\circ\kappa(\{1,\ldots,n\})\geqslant\tau\circ\kappa(l)$. Thus $\tau\circ\kappa(k)=\tau\circ\kappa(l)$ and all $\tau\circ\kappa(i)$ are equal. Condition 4.1(v) and induction on i,j then implies that $\tau\circ\kappa(\{i,\ldots,j\})$ are equal for all $1\leqslant i\leqslant j\leqslant n$. So neither of Definition 4.2(a),(b) apply for any i, as the strict inequalities do not hold. This proves (32).

4.3 An alternative formula for $S(*,\tau,\tilde\tau)$

We will need the following notation.

Definition 4.10. Let Condition 4.1 hold, and $(I,\preceq,\kappa)$ be $\mathcal{A}$-data. We say

- (i) $(I,\preceq,\kappa)$ is τ-*semistable* if $\tau(\kappa(J))\leqslant\tau(\kappa(I\setminus J))$ for all $(I,\preceq)$ s-sets $J\neq\emptyset,I$.
- (ii) $(I,\preceq,\kappa)$ is τ-*reversing* if $i\npreceq j$ implies $\tau\circ\kappa(i)<\tau\circ\kappa(j)$ for $i,j\in I$.

Part (i) is based on Definition 3.11, replacing subobjects by s-sets. If (σ,ι,π) is an $(I,\preceq,\kappa)$-configuration with $\sigma(I)$ τ-semistable, then Definition 3.11 implies $(I,\preceq,\kappa)$ is τ-semistable. In (ii), if $(I,\preceq,\kappa)$ is τ-reversing then $(I,\preceq)$ is a *total order*, and $i\mapsto\tau\circ\kappa(i)$ *reverses* the order of $(I,\preceq)$.

Proposition 4.11. *Suppose Condition 4.1 holds for $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$, and $(\{1,\ldots,n\},\leqslant,\kappa)$ is $\mathcal{A}$-data. Then*

$$S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)=\sum_{1\leqslant b\leqslant a\leqslant n}\;\sum_{\substack{\text{surjective }\alpha:\{1,\ldots,n\}\to\{1,\ldots,a\},\ \beta:\{1,\ldots,a\}\to\{1,\ldots,b\}:\\ 1\leqslant i\leqslant j\leqslant n\text{ implies }\alpha(i)\leqslant\alpha(j),\ 1\leqslant i\leqslant j\leqslant a\text{ implies }\beta(i)\leqslant\beta(j),\\ (\alpha^{-1}(j),\leqslant,\kappa)\ \tau\text{-reversing},\ j=1,\ldots,a.\ \text{Define }\nu:\{1,\ldots,b\}\to C(\mathcal{A})\\ \text{by }\nu(k)=\kappa((\beta\circ\alpha)^{-1}(k)).\ \text{Then }(\{1,\ldots,b\},\leqslant,\nu)\text{ is }\tilde\tau\text{-semistable}}}(-1)^{a-b}. \tag{38}$$

Proof. The τ-reversing, $\tilde\tau$-semistable conditions in (38) may be rewritten:

- $1 \leqslant i < n$ and $\alpha(i) = \alpha(i+1)$ implies $\tau \circ \kappa(i) > \tau \circ \kappa(i+1)$,
- $1 \leqslant i < n$ and $\beta\circ\alpha(i) \neq \beta\circ\alpha(i+1)$ implies $\tilde{\tau}\circ\kappa(\{1,\ldots,i\}) \leqslant \tilde{\tau}\circ\kappa(\{i+1,\ldots,n\})$.

Suppose first that each $1 \leqslant i < n$ satisfies Definition 4.2(a) or (b), and a, b, α, β are as in (38). If i satisfies (a) then $\alpha(i) \neq \alpha(i+1)$, so $\alpha(i+1) = \alpha(i) + 1$, and $\beta \circ \alpha(i+1) = \beta \circ \alpha(i)$. If i satisfies (b) there are three possibilities:

(i) $\alpha(i+1) = \alpha(i)$ and $\beta \circ \alpha(i+1) = \beta \circ \alpha(i)$,

(ii) $\alpha(i+1) = \alpha(i) + 1$ and $\beta \circ \alpha(i+1) = \beta \circ \alpha(i)$, or

(iii) $\alpha(i+1) = \alpha(i) + 1$ and $\beta \circ \alpha(i+1) = \beta \circ \alpha(i) + 1$.

Let r be the number of $i = 1, \ldots, n-1$ satisfying (a), and s_1, s_2, s_3 be the numbers of i satisfying (b) and (i), (ii) or (iii). Then $r + s_1 + s_2 + s_3 = n - 1$.

It is not difficult to see that for all of the $n - 1 - r$ values of i satisfying (b), any of (i)–(iii) is possible independently, and these choices of (i)–(iii) determine a, b, β, α uniquely. Hence there are 3^{n-1-r} possible quadruples a, b, α, β in (38). Furthermore, as $a - 1$ is the number of i with $\alpha(i+1) = \alpha(i) + 1$ and $b - 1$ the number of i with $\beta \circ \alpha(i+1) = \beta \circ \alpha(i) + 1$, we see that $a = 1 + r + s_2 + s_3$ and $b = 1 + s_3$, so that $(-1)^{a-b} = (-1)^r \cdot (-1)^{s_2}$. Therefore the sum over all 3^{n-1-r} possibilities of $(-1)^r \cdot (-1)^{s_2}$ equals $(-1)^r \cdot (1 - 1 + 1)^{n-1-r} = (-1)^r$, since $(1 - 1 + 1)^{n-1-r}$ is the sum over all $(n-1-r)$-tuples of choices of (i), (ii) or (iii) of the product $(-1)^{s_2}$ of 1 for each choice of (i), -1 for each (ii) and 1 for each (iii). Hence both sides of (38) are $(-1)^r$.

Now suppose some i does not satisfy Definition 4.2(a) or (b). Then either

(c) $\tau \circ \kappa(i) \leqslant \tau \circ \kappa(i+1)$ and $\tilde{\tau} \circ \kappa(\{1, \ldots, i\}) \leqslant \tilde{\tau} \circ \kappa(\{i+1, \ldots, n\})$, or

(d) $\tau \circ \kappa(i) > \tau \circ \kappa(i+1)$ and $\tilde{\tau} \circ \kappa(\{1, \ldots, i\}) > \tilde{\tau} \circ \kappa(\{i+1, \ldots, n\})$.

We shall show both sides of (38) are zero in each case.

Suppose i satisfies (c), and let a, b, α, β be as in (38). Then $\tau \circ \kappa(i) \leqslant \tau \circ \kappa(i+1)$ implies that $\alpha(i) \neq \alpha(i+1)$. Let $j = \alpha(i)$, so that $j + 1 = \alpha(i+1)$, and suppose $\beta(j) = \beta(j+1)$. Define $b' = b + 1$ and $\beta' : \{1, \ldots, a\} \to \{1, \ldots, b+1\}$ by $\beta'(k) = \beta(j)$ for $1 \leqslant k \leqslant j$, and $\beta'(k) = \beta(k) + 1$ for $j + 1 \leqslant k \leqslant a$. Then a, b', α, β' satisfy the conditions in (38), with $\beta'(j) \neq \beta'(j+1)$.

This establishes a 1-1 correspondence between quadruples (a, b, α, β) in (38) with $\beta \circ \alpha(i) = \beta \circ \alpha(i+1)$, and (a, b', α, β') in (38) with $\beta' \circ \alpha(i) \neq \beta' \circ \alpha(i+1)$. Each such pair contributes $(-1)^{a-b} + (-1)^{a-b'} = 0$ to (38), as $b' = b+1$, so both sides are zero. In the same way, if i satisfies (d) then $\beta \circ \alpha(i) = \beta \circ \alpha(i+1)$ for any a, b, α, β in (38). We construct a 1-1 correspondence between quadruples (a, b, α, β) in (38) with $\alpha(i) = \alpha(i+1)$ and (a', b, α', β') with $\alpha(i) \neq \alpha(i+1)$, where $a' = a + 1$. The contribution of each pair to (38) is zero, so both sides are zero. This completes the proof. □

4.4 Proof of equation (33)

We begin with two propositions.

Proposition 4.12. *Let Condition 4.1 hold, $(\{1,\ldots,b\},\leqslant,\mu)$ be $\mathcal{A}$-data, and $\gamma:\{1,\ldots,b\}\rightarrow\{1,\ldots,c\}$ be surjective with $i\leqslant j$ implies $\gamma(i)\leqslant\gamma(j)$. Then there exist unique $m=c,\ldots,b$ and surjective $\phi:\{1,\ldots,b\}\rightarrow\{1,\ldots,m\}$ and $\chi:\{1,\ldots,m\}\rightarrow\{1,\ldots,c\}$ with $\gamma=\chi\circ\phi$ and $i\leqslant j$ implies $\phi(i)\leqslant\phi(j)$ and $\chi(i)\leqslant\chi(j)$, such that if $\lambda:\{1,\ldots,m\}\rightarrow C(\mathcal{A})$ is given by $\lambda(i)=\mu\bigl(\phi^{-1}(i)\bigr)$ then $(\phi^{-1}(i),\leqslant,\mu)$ is τ-semistable for all $i=1,\ldots,m$, and $(\chi^{-1}(j),\leqslant,\lambda)$ is τ-reversing for all $j=1,\ldots,c$.*

Proof. First consider the case $c=1$. By induction choose a unique sequence $k_0,\ldots,k_m$ with $0=k_0<k_1<\cdots<k_m=b$, as follows. Set $k_0=0$. Having chosen k_i, if $k_i=b$ then put $m=i$ and finish. Otherwise, let k_{i+1} be largest such that $k_i<k_{i+1}\leqslant b$ and $(\{k_i+1,k_i+2,\ldots,k_{i+1}\},\leqslant,\mu)$ is τ-semistable. As $(\{k_i+1\},\leqslant,\mu)$ is τ-semistable, this is the maximum of a nonempty finite set, and k_{i+1} is well-defined. Now define $\phi:\{1,\ldots,b\}\rightarrow\{1,\ldots,m\}$ by $\phi(j)=i$ if $k_{i-1}<j\leqslant k_i$. Clearly ϕ is surjective with $1\leqslant i\leqslant j\leqslant b$ implies $\phi(i)\leqslant\phi(j)$. Define λ as above, and $\chi\equiv 1$.

By definition $\bigl(\phi^{-1}(i),\leqslant,\mu\bigr)=(\{k_{i-1}+1,\ldots,k_i\},\leqslant,\mu)$ is τ-*semistable* for all i, as we want. We shall show $(\{1,\ldots,m\},\leqslant,\lambda)$ is τ-*reversing*. Suppose for a contradiction that $1\leqslant i<m$ and $\tau\circ\lambda(i)\leqslant\tau\circ\lambda(i+1)$. This implies that

$$\tau\circ\mu(\{k_{i-1}+1,\ldots,k_i\})\leqslant\tau\circ\mu(\{k_{i-1}+1,\ldots,k_{i+1}\})\leqslant\tau\circ\mu(\{k_i+1,\ldots,k_{i+1}\}),\tag{39}$$

by Condition 4.1(v). Now $(\{k_{i-1}+1,\ldots,k_{i+1}\},\leqslant,\mu)$ is not τ-semistable by choice of k_i, so by Condition 4.1(v) there exists $k_{i-1}<j<k_{i+1}$ such that

$$\tau\circ\mu(\{k_{i-1}+1,\ldots,j\})\geqslant\tau\circ\mu(\{k_{i-1}+1,\ldots,k_{i+1}\})\geqslant\tau\circ\mu(\{j+1,\ldots,k_{i+1}\}),\tag{40}$$

with at least one of these inequalities *strict*. Divide into cases (i) $j=k_i$;

- (ii) $k_{i-1}<j<k_i$ and $\tau\circ\mu(\{k_{i-1}+1,\ldots,j\})>\tau\circ\mu(\{k_{i-1}+1,\ldots,k_{i+1}\})$;
- (iii) $k_{i-1}<j<k_i$ and $\tau\circ\mu(\{k_{i-1}+1,\ldots,k_{i+1}\})>\tau\circ\mu(\{j+1,\ldots,k_{i+1}\})$;
- (iv) $k_i<j<k_{i+1}$ and $\tau\circ\mu(\{k_{i-1}+1,\ldots,j\})>\tau\circ\mu(\{k_{i-1}+1,\ldots,k_{i+1}\})$;
- (v) $k_i<j<k_{i+1}$ and $\tau\circ\mu(\{k_{i-1}+1,\ldots,k_{i+1}\})>\tau\circ\mu(\{j+1,\ldots,k_{i+1}\})$.

In case (i) (39) and (40) are contradictory, as (40) has a strict inequality. In (ii) (39) gives $\tau\circ\mu(\{k_{i-1}+1,\ldots,j\})>\tau\circ\mu(\{k_{i-1}+1,\ldots,k_i\})$, implying $\tau\circ\mu(\{k_{i-1}+1,\ldots,j\})>\tau\circ\mu(\{j+1,\ldots,k_i\})$ by Condition 4.1(v), which contradicts $(\{k_{i-1}+1,\ldots,k_i\},\leqslant,\mu)$ τ-semistable. For (iii), as $\tau\circ\mu(\{j+1,\ldots,k_{i+1}\})$ lies between $\tau\circ\mu(\{j+1,\ldots,k_i\})$ and $\tau\circ\mu(\{k_i+1,\ldots,k_{i+1}\})$ by Condition 4.1(v) we have *either* $\tau\circ\mu(\{k_{i-1}+1,\ldots,k_{i+1}\})>\tau\circ\mu(\{j+1,\ldots,k_i\})$, giving $\tau\circ\mu(\{k_{i-1}+1,\ldots,j\})>\tau\circ\mu(\{j+1,\ldots,k_i\})$ by (40) contradicting $(\{k_{i-1}+1,\ldots,k_i\},\leqslant,\mu)$ τ-semistable, *or* $\tau\circ\mu(\{k_{i-1}+1,\ldots,k_{i+1}\})>\tau\circ\mu(\{k_i+1,\ldots,k_{i+1}\})$, contradicting (39). Similarly (iv),(v) give contradictions, using $(\{k_i+1,\ldots,k_{i+1}\},\leqslant,\mu)$ τ-semistable. Therefore $(\{1,\ldots,m\},\leqslant,\lambda)$ is τ-reversing.

This proves existence of m,ϕ,χ,λ. For uniqueness, suppose m',ϕ',χ',λ' also satisfy the conditions. Then $\phi'^{-1}(\{1,\ldots,i\})=\{1,\ldots,k_i'\}$ for unique $0=k_0'<k_1'<\cdots<k_m'=b$. As $(\{1,\ldots,k_1'\},\leqslant,\mu)$ is τ-semistable we have $k_1'\leqslant k_1$ by

choice of k_1. Suppose $k'_1 < k_1$. Then $\tau \circ \mu(\{1, \ldots, k_1\}) \geqslant \tau \circ \mu(\{1, \ldots, k'_1\})$ by Condition 4.1(v), as $1 \leqslant k'_1 < k_1$ and $(\{1, \ldots, k_1\}, \leqslant, \mu)$ is τ-semistable. Also

$$\tau \circ \mu(\{1, \ldots, k'_1\}) > \tau \circ \mu(\{k'_1 + 1, \ldots, k'_2\}) > \cdots > \tau \circ \mu(\{k'_{m-1} + 1, \ldots, k'_m\})$$

as $(\{1, \ldots, m'\}, \leqslant, \lambda')$ is τ-reversing. Now $k'_i < k_1 \leqslant k'_{i+1}$ for some $1 \leqslant i < m'$, and using the inequalities above and Condition 4.1(v) gives $\tau \circ \mu(\{1, \ldots, k_1\}) > \tau \circ \mu(\{1, \ldots, k'_i\})$, so that

$$\tau \circ \mu(\{k'_i + 1, \ldots, k_1\}) > \tau \circ \mu(\{1, \ldots, k'_i\}) > \tau \circ \mu(\{k'_i + 1, \ldots, k'_{i+1}\}),$$

contradicting both $k_1 = k'_{i+1}$, and τ-semistability of $\bigl(\{k'_i + 1, \ldots, k'_{i+1}\}, \leqslant, \mu\bigr)$ if $k_1 < k'_{i+1}$ by Condition 4.1(v). Therefore $k'_1 = k_1$. Extending this argument shows $k'_i = k_i$ for $i = 1, 2, \ldots$ and $m = m'$, proving uniqueness of m, ϕ, χ, and the proposition for $c = 1$.

For $c > 1$ we apply the $c = 1$ case to $(\gamma^{-1}(\{j\}), \leqslant, \mu)$ for $j = 1, \ldots, c$. This gives unique m_j, $\phi_j : \gamma^{-1}(\{j\}) \to \{1, \ldots, m_j\}$ and λ_j such that $(\phi_j^{-1}(i), \leqslant, \mu)$ is τ-semistable for $i = 1, \ldots, m_j$, and $(\{1, \ldots, m_j\}, \leqslant, \lambda_j)$ is τ-reversing for $j = 1, \ldots, c$. Define $m = m_1 + \cdots + m_c$ and $\phi : \{1, \ldots, b\} \to \{1, \ldots, m\}$ by $\phi(i) = m_1 + \cdots + m_{j-1} + \phi_j(i)$ when $\gamma(i) = j$, and $\chi : \{1, \ldots, m\} \to \{1, \ldots, c\}$ by $\chi(i) = j$ when $m_1 + \cdots + m_{j-1} < i \leqslant m_1 + \cdots + m_j$. Then m, ϕ, χ satisfy the proposition. Uniqueness of m, ϕ, χ follows easily from that of the m_j, ϕ_j. $\square$

Proposition 4.13. *Let $1 \leqslant d \leqslant b$ and $\epsilon : \{1, \ldots, b\} \to \{1, \ldots, d\}$ be surjective with $1 \leqslant i \leqslant j \leqslant b$ implies $\epsilon(i) \leqslant \epsilon(j)$. Then*

$$\sum_{c=d}^{b} \sum_{\substack{\text{surjective } \gamma : \{1, \ldots, b\} \to \{1, \ldots, c\} \text{ and} \\ \delta : \{1, \ldots, c\} \to \{1, \ldots, d\} \text{ with } \epsilon = \delta \circ \gamma: \\ i \leqslant j \text{ implies } \gamma(i) \leqslant \gamma(j),\ i \leqslant j \text{ implies } \delta(i) \leqslant \delta(j)}} (-1)^{b-c} = \begin{cases} 1, & d = b, \\ 0, & \text{otherwise.} \end{cases} \tag{41}$$

Proof. The first line of (41) is immediate as if $d = b$ the only possibility is $c = b$ and $\epsilon = \gamma = \delta = \mathrm{id}$. So suppose $d < b$. Define $S = \{i = 1, \ldots, b - 1 : \epsilon(i + 1) = \epsilon(i)\}$. Then $|S| = b - d$. For γ, δ as in (41), define $T = \{i \in S : \gamma(i + 1) = \gamma(i)\}$. Then $|T| = c - d$, and T determines γ by $\gamma(1) = 1$, $\gamma(i + 1) = \gamma(i)$ if $i \in T$ and $\gamma(i + 1) = \gamma(i) + 1$ if $i \in \{1, \ldots, b - 1\} \setminus T$. Also c, ϵ, γ determine δ by $\epsilon = \delta \circ \gamma$. This establishes a 1-1 correspondence between choices of c, γ, δ in (41) and subsets $T \subseteq S$ with $|T| = c - d$. Hence the l.h.s. of (41) becomes

$$\sum_{\text{subsets } T \subseteq S} (-1)^{|S \setminus T|} = \sum_{j=0}^{b-d} \binom{b-d}{j} (-1)^{b-d-j} = (1 - 1)^{b-d} = 0,$$

as there are $\binom{b-d}{j}$ subsets $T \subseteq S$ with $|T| = j$. $\square$

To complete the proof of Theorem 4.5 we use sets and maps as follows:

$$\begin{tikzcd}
\{1, \ldots, n\} \arrow[r, "\alpha"] \arrow[rrr, bend right, "\psi"'] & \{1, \ldots, a\} \arrow[r, "\beta"] & \{1, \ldots, b\} \arrow[r, "\phi"] \arrow[rr, bend left, "\gamma"] & \{1, \ldots, m\} \arrow[r, "\chi"] \arrow[rr, bend right, "\epsilon"'] & \{1, \ldots, c\} \arrow[r, "\delta"] & \{1, \ldots, d\},
\end{tikzcd}$$

and maps $\kappa:\{1,\ldots,n\}\rightarrow C(\mathcal{A})$, $\lambda:\{1,\ldots,m\}\rightarrow C(\mathcal{A})$, $\mu:\{1,\ldots,b\}\rightarrow C(\mathcal{A})$ and $\nu:\{1,\ldots,d\}\rightarrow C(\mathcal{A})$. We deduce (33) from Propositions 4.11–4.13 by:

$$
\begin{aligned}
&\sum_{m=1}^{n}\;\sum_{\substack{\psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}:\\ \psi\text{ is surjective,}\\ 1\leqslant i\leqslant j\leqslant n\text{ implies }\psi(i)\leqslant\psi(j),\\ \text{define }\lambda:\{1,\ldots,m\}\rightarrow C(\mathcal{A})\\ \text{by }\lambda(k)=\kappa(\psi^{-1}(k))}} \begin{array}{l} S(\{1,\ldots,m\},\leqslant,\lambda,\hat\tau,\tilde\tau)\cdot\\ \prod_{k=1}^{m} S\bigl(\psi^{-1}(\{k\}),\leqslant,\kappa|_{\psi^{-1}(\{k\})},\tau,\hat\tau\bigr)= \end{array}\\
&\sum_{\substack{m=1,\ldots,n\text{ and}\\ \psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}:\\ \psi\text{ is surjective,}\\ i\leqslant j\text{ implies }\psi(i)\leqslant\psi(j),\\ \text{define }\lambda:\{1,\ldots,m\}\rightarrow C(\mathcal{A})\\ \text{by }\lambda(k)=\kappa(\psi^{-1}(k))}} \left[\sum_{1\leqslant d\leqslant c\leqslant m}\;\sum_{\substack{\text{surjective }\chi:\{1,\ldots,m\}\rightarrow\{1,\ldots,c\}\\ \text{and }\delta:\{1,\ldots,c\}\rightarrow\{1,\ldots,d\}:\\ i\leqslant j\text{ implies }\chi(i)\leqslant\chi(j),\; i\leqslant j\text{ implies }\delta(i)\leqslant\delta(j),\\ (\chi^{-1}(j),\leqslant,\lambda)\;\hat\tau\text{-reversing, }1\leqslant j\leqslant c.\\ \text{Let }\nu:\{1,\ldots,d\}\rightarrow C(\mathcal{A})\text{ be }\nu:k\mapsto\lambda((\delta\circ\chi)^{-1}(k)).\\ \text{Then }(\{1,\ldots,d\},\leqslant,\nu)\text{ is }\tilde\tau\text{-semistable}}} (-1)^{c-d}\right]\cdot\\
&\left[\sum_{m\leqslant b\leqslant a\leqslant n}\;\sum_{\substack{\text{surjective }\alpha:\{1,\ldots,n\}\rightarrow\{1,\ldots,a\},\;\beta:\{1,\ldots,a\}\rightarrow\{1,\ldots,b\}\\ \text{and }\phi:\{1,\ldots,b\}\rightarrow\{1,\ldots,m\}\text{ with }\psi=\phi\circ\beta\circ\alpha:\\ i\leqslant j\text{ implies }\alpha(i)\leqslant\alpha(j),\; i\leqslant j\text{ implies }\beta(i)\leqslant\beta(j),\\ i\leqslant j\text{ implies }\phi(i)\leqslant\phi(j),\;(\alpha^{-1}(j),\leqslant,\kappa)\;\tau\text{-reversing, }1\leqslant j\leqslant a.\\ \text{Let }\mu:\{1,\ldots,b\}\rightarrow C(\mathcal{A})\text{ be }\mu:k\mapsto\kappa((\beta\circ\alpha)^{-1}(k)).\\ \text{Then }(\phi^{-1}(i),\leqslant,\mu)\text{ is }\hat\tau\text{-semistable, }1\leqslant i\leqslant m}} (-1)^{a-b}\right]=\\
&\sum_{d\leqslant b\leqslant a\leqslant n}\;\sum_{\substack{\text{surjective }\alpha:\{1,\ldots,n\}\rightarrow\{1,\ldots,a\},\\ \beta:\{1,\ldots,a\}\rightarrow\{1,\ldots,b\}\text{ and }\epsilon:\{1,\ldots,b\}\rightarrow\{1,\ldots,d\}:\\ i\leqslant j\text{ implies }\alpha(i)\leqslant\alpha(j),\; i\leqslant j\text{ implies }\beta(i)\leqslant\beta(j),\\ i\leqslant j\text{ implies }\epsilon(i)\leqslant\epsilon(j),\;(\alpha^{-1}(j),\leqslant,\kappa)\;\tau\text{-reversing, }1\leqslant j\leqslant a.\\ \text{Let }\nu:\{1,\ldots,d\}\rightarrow C(\mathcal{A})\text{ be }\nu:k\mapsto\kappa((\epsilon\circ\beta\circ\alpha)^{-1}(k)).\\ \text{Then }(\{1,\ldots,d\},\leqslant,\nu)\text{ is }\tilde\tau\text{-semistable}}} (-1)^{a-d}\cdot\\
&\left[\sum_{d\leqslant c\leqslant b}\;\sum_{\substack{\text{surjective }\gamma:\{1,\ldots,b\}\rightarrow\{1,\ldots,c\}\text{ and}\\ \delta:\{1,\ldots,c\}\rightarrow\{1,\ldots,d\}\text{ with }\epsilon=\delta\circ\gamma:\\ i\leqslant j\text{ implies }\gamma(i)\leqslant\gamma(j),\; i\leqslant j\text{ implies }\delta(i)\leqslant\delta(j)}} (-1)^{b-c}\right]\\
&=S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau).
\end{aligned}
\tag{42}
$$

Here the term $[\cdots]$ on the second line is $S(\{1,\ldots,m\},\leqslant,\lambda,\hat\tau,\tilde\tau)$ by (38), and $[\cdots]$ on the third line is $\prod_{k=1}^{m}S\bigl(\psi^{-1}(\{k\}),\leqslant,\kappa|_{\psi^{-1}(\{k\})},\tau,\hat\tau\bigr)$ by (38), where we have relabelled and combined the product over $k=1,\ldots,m$ of sums over a_k, $\alpha_k:\psi^{-1}(\{k\})\rightarrow\{1,\ldots,a_k\}$ and b_k, $\beta_k:\{1,\ldots,a_k\}\rightarrow\{1,\ldots,b_k\}$ into one large sum over a,b,α,β and ϕ, where $a=a_1+\cdots+a_m$ and $b=b_1+\cdots+b_m$.

To deduce the fourth and fifth lines of (42) from the second and third, we set $\gamma=\chi\circ\phi$. Then the conditions $i\leqslant j$ implies $\chi(i)\leqslant\chi(j)$ and $(\chi^{-1}(j),\leqslant,\lambda)$ τ-reversing for $j=1,\ldots,c$ in the second line, and $i\leqslant j$ implies $\phi(i)\leqslant\phi(j)$, $(\phi^{-1}(i),\leqslant,\mu)$ $\hat\tau$-semistable for $i=1,\ldots,m$ in the third line, are the hypotheses of Proposition 4.12 with $\hat\tau$ in place of τ. Thus Proposition 4.12 shows that for all choices of b,c,γ and μ there are unique choices of m,ϕ,χ satisfying the conditions involving $\hat\tau$. So we drop the sums over m,ϕ,χ,ψ and the $\hat\tau$ conditions, and also rearrange the sums.

To deduce the sixth and last line of (42), note that the term $[\cdots]$ on the fifth line is 1 if $d=b$ and 0 otherwise, by Proposition 4.13. If $d=b$ then ϵ is the identity, and the fourth line of (42) reduces to $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)$ by (38). This completes the proofs of (33) and Theorem 4.5.

5 Transforming between stability conditions

We now prove the transformation laws (27)–(29) from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$, and their stack function analogues. For most of the section we do not suppose $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ are *permissible*. Therefore our equations are identities in $\dot{\mathrm{LCF}},\dot{\mathrm{LSF}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal A})$ rather than $\mathrm{CF},\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal A})$, which may have *infinitely many nonzero terms*, and must be interpreted using a notion of *convergence*.

This is partly for greater generality, but mostly because even when $(\tau,T,\leqslant)$, $(\tilde\tau,\tilde T,\leqslant)$ are permissible, to prove (27)–(29) we may need to go via an intermediate weak stability condition $(\hat\tau,\hat T,\leqslant)$ which is not permissible, which happens when $\mathcal A=\mathrm{coh}(P)$. It is far from obvious that (27)–(29) have only finitely many nonzero terms even for permissible $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$. We prove this for $\mathcal A=\mathrm{coh}(P)$ when P is a smooth surface and $\tau,\tilde\tau$ Gieseker stability conditions.

The (locally) constructible functions material below needs the ground field $\mathbb K$ to have *characteristic zero*, but the (local) stack function versions work for $\mathbb K$ of *arbitrary characteristic*. As we develop the two strands in parallel, for brevity we make the convention that $\mathrm{char}\,\mathbb K=0$ in the parts dealing with $\mathrm{CF},\dot{\mathrm{LCF}}(\mathfrak{Obj}_{\mathcal A})$ and $\mathrm{char}\,\mathbb K$ is arbitrary otherwise, and will mostly not remark on it.

5.1 Basic definitions and main results

We will need the following finiteness conditions.

Definition 5.1. Let Assumption 3.5 hold and $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be weak stability conditions on $\mathcal A$. We say *the change from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ is locally finite* if for all constructible $U\subseteq\mathfrak{Obj}_{\mathcal A}(\mathbb K)$, there are only finitely many sets of $\mathcal A$-data $(\{1,\ldots,n\},\leqslant,\kappa)$ for which $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\neq 0$ and

$$U\cap\boldsymbol\sigma(\{1,\ldots,n\})_*\big(\mathcal M_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal A}\big)\neq\emptyset. \tag{43}$$

We say *the change from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ is globally finite* if this holds for $U=\mathfrak{Obj}^\alpha_{\mathcal A}(\mathbb K)$ (which is not constructible, in general) for all $\alpha\in C(\mathcal A)$. Since any constructible $U\subseteq\mathfrak{Obj}_{\mathcal A}(\mathbb K)$ is contained in a finite union of $\mathfrak{Obj}^\alpha_{\mathcal A}(\mathbb K)$, globally finite implies locally finite.

When we say *the changes between $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$*, we mean both the change from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ and the change from $(\tilde\tau,\tilde T,\leqslant)$ to $(\tau,T,\leqslant)$.

Here is the main result of this section.

Theorem 5.2. *Let Assumption 3.5 hold and $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant),(\hat\tau,\hat T,\leqslant)$ be weak stability conditions on $\mathcal A$ with $(\hat\tau,\hat T,\leqslant)$ dominating $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$, and*

suppose the change from $(\hat\tau,\hat T,\leqslant)$ *to* $(\tilde\tau,\tilde T,\leqslant)$ *is locally finite. Then the change from* $(\tau,T,\leqslant)$ *to* $(\tilde\tau,\tilde T,\leqslant)$ *is locally finite.*

Let $\alpha\in C(\mathcal{A})$ *and* $(K,\trianglelefteq,\mu)$ *be* $\mathcal{A}$*-data. Then*

$$\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\cdot\\ &\quad\delta_{\rm ss}^{\kappa(1)}(\tau)*\delta_{\rm ss}^{\kappa(2)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa(n)}(\tau)=\delta_{\rm ss}^{\alpha}(\tilde\tau),\end{aligned} \tag{44}$$

$$\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\cdot\\ &\quad\bar\delta_{\rm ss}^{\kappa(1)}(\tau)*\bar\delta_{\rm ss}^{\kappa(2)}(\tau)*\cdots*\bar\delta_{\rm ss}^{\kappa(n)}(\tau)=\bar\delta_{\rm ss}^{\alpha}(\tilde\tau),\end{aligned} \tag{45}$$

$$\sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ (I,\preceq,K,\phi)\text{ is dominant},\\ \trianglelefteq=\mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k)\text{ for }k\in K}} \begin{aligned}[t] &T(I,\preceq,\kappa,K,\phi,\tau,\tilde\tau)\cdot\\ &\mathrm{CF}^{\rm stk}\bigl(Q(I,\preceq,K,\trianglelefteq,\phi)\bigr)\\ &\quad\delta_{\rm ss}(I,\preceq,\kappa,\tau)=\delta_{\rm ss}(K,\trianglelefteq,\mu,\tilde\tau),\end{aligned} \tag{46}$$

$$\sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ (I,\preceq,K,\phi)\text{ is dominant},\\ \trianglelefteq=\mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k)\text{ for }k\in K}} \begin{aligned}[t] &T(I,\preceq,\kappa,K,\phi,\tau,\tilde\tau)\cdot\\ &Q(I,\preceq,K,\trianglelefteq,\phi)_*\\ &\quad\bar\delta_{\rm ss}(I,\preceq,\kappa,\tau)=\bar\delta_{\rm ss}(K,\trianglelefteq,\mu,\tilde\tau).\end{aligned} \tag{47}$$

If also $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ *are permissible then*

$$\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\cdot\\ &\epsilon^{\kappa(1)}(\tau)*\epsilon^{\kappa(2)}(\tau)*\cdots*\epsilon^{\kappa(n)}(\tau)=\epsilon^{\alpha}(\tilde\tau),\end{aligned} \tag{48}$$

$$\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\cdot\\ &\bar\epsilon^{\kappa(1)}(\tau)*\bar\epsilon^{\kappa(2)}(\tau)*\cdots*\bar\epsilon^{\kappa(n)}(\tau)=\bar\epsilon^{\alpha}(\tilde\tau).\end{aligned} \tag{49}$$

Here (44)–(49) *are equations in* $\mathrm{L\dot{C}F},\mathrm{L\dot{S}F}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}}),\mathrm{LCF},\mathrm{LSF}(\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}})$, $\mathrm{CF},\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ *respectively, with* $\operatorname{char}\mathbb{K}=0$ *in* (44)*,* (46)*,* (48)*. There may be infinitely many nonzero terms on the left hand sides of* (44)–(49)*, but they hold as convergent sums in* $\mathrm{LCF},\mathrm{LSF}(\cdots)$ *in the sense of Definition 2.16.*

If the change from $(\tau,T,\leqslant)$ *to* $(\tilde\tau,\tilde T,\leqslant)$ *is globally finite then there are only finitely many nonzero terms in* (44)–(49)*, and they hold as finite sums.*

We assume $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ *permissible* in (48)–(49), as we defined $\epsilon^\alpha(\tau)$, $\bar\epsilon^\alpha(\tau)$ this way in [31, §7–§8]. But (48)–(49) also hold in $\mathrm{L\dot{C}F},\mathrm{L\dot{S}F}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ if $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ are stability conditions and $\mathcal{A}$ is τ- and $\tilde\tau$-artinian.

Suppose (48) holds as a *finite* sum. Then $\epsilon^\alpha(\tilde\tau)$ lies in $\mathcal{H}^{\rm to}_\tau$ and $\mathrm{CF}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})$, so by Definition 3.19 it lies in the Lie algebra $\mathcal{L}^{\rm to}_\tau$, which is generated by the $\epsilon^\beta(\tau)$. That is, $\epsilon^\alpha(\tilde\tau)$ is a $\mathbb{Q}$-linear combination of multiple commutators such as

$$[\,[\cdots[\,[\epsilon^{\kappa(1)}(\tau),\epsilon^{\kappa(2)}(\tau)],\epsilon^{\kappa(3)}(\tau)],\cdots],\epsilon^{\kappa(n)}(\tau)].$$

Thus it is natural to ask whether we can rewrite (48) as a sum of commutators of $\epsilon^{\kappa(i)}(\tau)$ rather than a sum of products, so that it becomes an identity in the

Lie algebra $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ rather than the algebra $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$. The next theorem shows that we can. First we introduce some notation.

Definition 5.3. For I a finite set, write $\mathbb{Q}[I]$ for the associative, noncommutative $\mathbb{Q}$-algebra generated by $i \in I$. That is, elements of $\mathbb{Q}[I]$ are finite linear combinations of *words* $i_1 i_2 \cdots i_m$ for $m \geqslant 0$ and $i_1, \ldots, i_m \in I$, with multiplication given by juxtaposition. If $J \subseteq I$ and $\preceq$ is a total order on J, write

$$\prod_{j \in J \text{ in order } \preceq} j \;= j_1 j_2 \cdots j_n \text{ if } J = \{j_1, \ldots, j_n\} \text{ with } j_1 \preceq j_2 \preceq \cdots \preceq j_n.$$

We can regard $\mathbb{Q}[I]$ as a Lie algebra, with commutator $[i_1 \cdots i_m, j_1 \cdots j_n] = i_1 \cdots i_m j_1 \cdots j_n - j_1 \cdots j_n i_1 \cdots i_m$. Define $\mathcal{L}[I]$ to be the Lie subalgebra of $\mathbb{Q}[I]$ generated by $i \in I$. Then $\mathcal{L}[I]$ is the free $\mathbb{Q}$-Lie algebra generated by I, and $\mathbb{Q}[I]$ is naturally isomorphic to $U(\mathcal{L}[I])$.

Theorem 5.4. *Let Condition 4.1 hold for $(\tau, T, \leqslant)$ and $(\tilde\tau, \tilde T, \leqslant)$, and I be a finite set and $\kappa : I \to C(\mathcal{A})$. Then the expression in $\mathbb{Q}[I]$*

$$\sum_{\text{total orders } \preceq \text{ on } I} U(I, \preceq, \kappa, \tau, \tilde\tau) \cdot \prod_{i \in I \text{ in order } \preceq} i \qquad \text{lies in } \mathcal{L}[I]. \tag{50}$$

The author has a purely combinatorial proof of Theorem 5.4, but it is long and complicated, so we instead give a shorter constructible functions proof. By an easy combinatorial argument (48)–(49) may be rewritten

$$\epsilon^\alpha(\tilde\tau) = \sum_{\substack{\text{iso classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!} \sum_{\substack{\kappa: I \to C(\mathcal{A}):\\ \kappa(I) = \alpha}} \left[\sum_{\substack{\text{total orders } \preceq \text{ on } I.\\ \text{Write } I = \{i_1, \ldots, i_n\},\\ i_1 \preceq i_2 \preceq \cdots \preceq i_n}} \begin{matrix} U(I, \preceq, \kappa, \tau, \tilde\tau) \cdot \\ \epsilon^{\kappa(i_1)}(\tau) * \cdots * \epsilon^{\kappa(i_n)}(\tau) \end{matrix} \right], \tag{51}$$

$$\bar\epsilon^\alpha(\tilde\tau) = \sum_{\substack{\text{iso classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!} \sum_{\substack{\kappa: I \to C(\mathcal{A}):\\ \kappa(I) = \alpha}} \left[\sum_{\substack{\text{total orders } \preceq \text{ on } I.\\ \text{Write } I = \{i_1, \ldots, i_n\},\\ i_1 \preceq i_2 \preceq \cdots \preceq i_n}} \begin{matrix} U(I, \preceq, \kappa, \tau, \tilde\tau) \cdot \\ \bar\epsilon^{\kappa(i_1)}(\tau) * \cdots * \bar\epsilon^{\kappa(i_n)}(\tau) \end{matrix} \right]. \tag{52}$$

By Theorem 5.4, the terms $[\cdots]$ are linear combinations of commutators of $\epsilon^{\kappa(i)}, \bar\epsilon^{\kappa(i)}$ for $i \in I$, so (51)–(52) are identities in the Lie algebras $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$. We can now deduce:

Corollary 5.5. *Let Assumption 3.5 hold and $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant), (\hat\tau, \hat T, \leqslant)$ be weak stability conditions on $\mathcal{A}$ with $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$ permissible and $(\hat\tau, \hat T, \leqslant)$ dominating $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$. Suppose the changes from $(\hat\tau, \hat T, \leqslant)$ to $(\tau, T, \leqslant)$, $(\tilde\tau, \tilde T, \leqslant)$ are locally finite, and the changes between $(\tau, T, \leqslant)$ and $(\tilde\tau, \tilde T, \leqslant)$ are globally finite. Then in the notation of §3.5 we have*

$$\begin{gathered} \mathcal{H}^{\mathrm{pa}}_\tau = \mathcal{H}^{\mathrm{pa}}_{\tilde\tau}, \quad \mathcal{H}^{\mathrm{to}}_\tau = \mathcal{H}^{\mathrm{to}}_{\tilde\tau}, \quad \bar{\mathcal{H}}^{\mathrm{pa}}_\tau = \bar{\mathcal{H}}^{\mathrm{pa}}_{\tilde\tau}, \quad \bar{\mathcal{H}}^{\mathrm{to}}_\tau = \bar{\mathcal{H}}^{\mathrm{to}}_{\tilde\tau}, \\ \mathcal{L}^{\mathrm{pa}}_\tau = \mathcal{L}^{\mathrm{pa}}_{\tilde\tau}, \quad \mathcal{L}^{\mathrm{to}}_\tau = \mathcal{L}^{\mathrm{to}}_{\tilde\tau}, \quad \bar{\mathcal{L}}^{\mathrm{pa}}_\tau = \bar{\mathcal{L}}^{\mathrm{pa}}_{\tilde\tau}, \quad \bar{\mathcal{L}}^{\mathrm{to}}_\tau = \bar{\mathcal{L}}^{\mathrm{to}}_{\tilde\tau}. \end{gathered} \tag{53}$$

Proof. Theorem 5.2 implies (44)–(49) hold as finite sums in $\mathrm{CF},\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{CF},\mathrm{SF}(\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}})$. Thus $\delta_{\mathrm{ss}}^{\alpha}(\tilde\tau)\in\mathcal{H}_{\tau}^{\mathrm{to}}$, $\bar\delta_{\mathrm{ss}}^{\alpha}(\tilde\tau)\in\bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$, and applying $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(K)),\boldsymbol{\sigma}(K)_*$ to (46)–(47) gives $\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(K))\delta_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tilde\tau)\in\mathcal{H}_{\tau}^{\mathrm{pa}}$ and $\boldsymbol{\sigma}(K)_*\,\bar\delta_{\mathrm{ss}}(K,\trianglelefteq,\mu,\tilde\tau)\in\bar{\mathcal{H}}_{\tau}^{\mathrm{pa}}$. Hence $\mathcal{H}_{\tilde\tau}^{\mathrm{pa}}\subseteq\mathcal{H}_{\tau}^{\mathrm{pa}},\ldots,\bar{\mathcal{H}}_{\tilde\tau}^{\mathrm{to}}\subseteq\bar{\mathcal{H}}_{\tau}^{\mathrm{to}}$. Exchanging $\tau,\tilde\tau$ proves the top line of (53). The first three equations of the bottom line follow from Definition 3.19. For the last, (52) and Theorem 5.4 imply $\bar\epsilon^{\alpha}(\tilde\tau)\in\bar{\mathcal{L}}_{\tau}^{\mathrm{to}}$, so $\bar{\mathcal{L}}_{\tilde\tau}^{\mathrm{to}}\subseteq\bar{\mathcal{L}}_{\tau}^{\mathrm{to}}$, and exchanging $\tau,\tilde\tau$ gives the result. $\square$

In [29, Ex.s 10.5–10.9] we define data $\mathcal{A},K(\mathcal{A}),\mathfrak{F}_{\mathcal{A}}$ satisfying Assumption 3.5 with $\mathcal{A}=\text{mod-}\mathbb{K}Q$ or $\text{nil-}\mathbb{K}Q$ for $Q=(Q_0,Q_1,b,e)$ a *quiver*, and $\mathcal{A}=\text{mod-}\mathbb{K}Q/I$ or $\text{nil-}\mathbb{K}Q/I$ for (Q,I) a *quiver with relations*, and $\mathcal{A}=\text{mod-}A$ for A a *finite-dimensional* $\mathbb{K}$*-algebra*. The last case $\mathcal{A}=\text{mod-}A$ also has an associated quiver, the *Ext-quiver* Q of A, and $K(\mathcal{A})\cong\mathbb{Z}^{Q_0}$ in each case. All weak stability conditions on these $\mathcal{A}$ are permissible on by [31, Cor. 4.13], and as for each $\alpha\in C(\mathcal{A})$ there are only finitely many $\beta,\gamma\in C(\mathcal{A})$ with $\alpha=\beta+\gamma$, the changes between any two weak stability conditions are globally finite.

For any such $\mathcal{A}$ we have the trivial stability condition $(0,\{0\},\leqslant)$, mapping $\alpha\mapsto 0$ for all $\alpha\in C(\mathcal{A})$. This dominates all weak stability conditions on $\mathcal{A}$. So applying Theorem 5.2 with $(\hat\tau,\hat T,\leqslant)=(0,\{0\},\leqslant)$ and Corollary 5.5 gives:

Corollary 5.6. *Let $\mathcal{A},K(\mathcal{A}),\mathfrak{F}_{\mathcal{A}}$ be as in one of the quiver examples* [29, Ex.s 10.5–10.9]*, and let $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be any two weak stability conditions on $\mathcal{A}$, such as the slope functions of* [31, Ex. 4.14]. *Then $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ are permissible, and the changes between them are globally finite, and* (44)–(49) *hold as finite sums in* $\mathrm{CF},\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ *or* $\mathrm{CF},\mathrm{SF}(\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}})$*, and* (53) *holds. Thus $\mathcal{H}_{\tau}^{\mathrm{pa}},\ldots,\bar{\mathcal{L}}_{\tau}^{\mathrm{to}}$ are independent of the weak stability condition $(\tau,T,\leqslant)$.*

In [31, §4.4] we studied weak stability conditions on $\mathcal{A}=\mathrm{coh}(P)$, the abelian category of *coherent sheaves* on a projective $\mathbb{K}$-scheme P, with data $K(\mathrm{coh}(P)),\mathfrak{F}_{\mathrm{coh}(P)}$ as in [29, Ex. 9.1 or 9.2]. With $m=\dim P$, we defined [31, Ex.s 4.16–4.18] a permissible stability condition $(\gamma,G_m,\leqslant)$ from *Gieseker stability*, a permissible weak stability condition $(\mu,M_m,\leqslant)$ from *μ-stability*, and a non-permissible weak stability condition $(\delta,D_m,\leqslant)$ from *purity* and the *torsion filtration*, which dominates $(\gamma,G_m,\leqslant)$ and $(\mu,M_m,\leqslant)$.

Proposition 5.7. *In the situation of* [31, §4.4] *with $\mathcal{A}=\mathrm{coh}(P)$, the changes from $(\delta,D_m,\leqslant)$ to $(\gamma,G_m,\leqslant),(\mu,M_m,\leqslant)$ are locally finite.*

Proof. We first verify Definition 5.1 with $(\tau,T,\leqslant)=(\delta,D_m,\leqslant)$, $(\tilde\tau,\tilde T,\leqslant)=(\gamma,G_m,\leqslant)$ and $U=\{[X]\}$ with $[X]=\alpha\in C(\mathcal{A})$. If $(\{1,\ldots,n\},\leqslant,\kappa)$ is $\mathcal{A}$-data with $\kappa(\{1,\ldots,n\})=\alpha$ then the argument of (60) below shows that

$$S(\{1,\ldots,n\},\leqslant,\kappa,\delta,\gamma)=\begin{cases}(-1)^{n-1}, & \begin{array}{l}\gamma\circ\kappa(\{1,\ldots,i\})>\gamma\circ\kappa(\{i+1,\ldots,n\})\\ \text{for all } 1\leqslant i<n, \text{ and } \delta\circ\kappa\equiv\delta(\alpha),\end{array}\\ 0, & \text{otherwise.}\end{cases}$$

Suppose $S(\{1,\ldots,n\},\leqslant,\kappa,\delta,\gamma)\neq 0$ and $[(\sigma,\iota,\pi)]\in\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\delta)_{\mathcal{A}}$ with $\sigma(\{1,\ldots,n\})=X$. As the $\sigma(\{i\})$ are δ-semistable and $\delta\equiv\delta(\alpha)=\delta([X])$, by [31, Ex. 4.18] the $\sigma(\{i\})$ are all *pure* with $\dim\sigma(\{i\})=\dim X$.

By induction we see that $\sigma(\{i+1,\ldots,n\})$ is pure of dimension $\dim X$ for all $1 \leqslant i < n$. But we have an exact sequence

$$0 \longrightarrow \sigma(\{1,\ldots,i\}) \xrightarrow{\iota(\{1,\ldots,i\},\{1,\ldots,n\})} X \xrightarrow{\pi(\{1,\ldots,n\},\{i+1,\ldots,n\})} \sigma(\{i+1,\ldots,n\}) \longrightarrow 0,$$

so $\sigma(\{i+1,\ldots,n\})$ is a *quotient sheaf* of X. Also, $\gamma \circ \kappa(\{1,\ldots,i\}) > \gamma \circ \kappa(\{i+1,\ldots,n\})$ implies $\gamma(\alpha) \geqslant \gamma \circ \kappa(\{i+1,\ldots,n\})$ by the weak seesaw inequality, and as $\deg \gamma(\alpha) = \deg \gamma \circ \kappa(\{i+1,\ldots,n\})$ this implies that $\hat{\mu}\bigl(\sigma(\{i+1,\ldots,n\})\bigr) \leqslant \hat{\mu}(X)$, where $\hat{\mu}$ is the *slope* of [23, Def. 1.6.8], basically the second coefficient in the γ polynomial.

Now Huybrechts and Lehn [23, Lem. 1.7.9 & Rem. 1.7.10] prove that if $X \in \mathrm{coh}(P)$ if fixed and $\mu_0 \in \mathbb{R}$, the family of all pure quotient sheaves Y of X with $\dim Y = \dim X$ and $\hat{\mu}([Y]) \leqslant \mu_0$ is constructible, and hence realizes only finitely many classes in $C(\mathcal{A})$. Applying this with $\mu_0 = \hat{\mu}(X)$ shows there are only finitely many possibilities for $\kappa(\{i+1,\ldots,n\})$ in $C(\mathcal{A})$. It easily follows that there are only finitely many possibilities for n, κ, as we want.

This extends to arbitrary constructible $U \subseteq \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ using a families version of [23, Lem. 1.7.9 & Rem. 1.7.10], so the change from $(\delta, D_m, \leqslant)$ to $(\gamma, G_m, \leqslant)$ is locally finite. The proof for $(\mu, M_m, \leqslant)$ is the same, since $\mu(\alpha) \geqslant \mu \circ \kappa(\{i+1,\ldots,n\})$ and $\deg \mu(\alpha) = \deg \mu \circ \kappa(\{i+1,\ldots,n\})$ also imply $\hat{\mu}\bigl(\sigma(\{i+1,\ldots,n\})\bigr) \leqslant \hat{\mu}(X)$. $\square$

In Theorem 5.2 choose $(\hat{\tau}, \hat{T}, \leqslant) = (\delta, D_m, \leqslant)$ and $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ arbitrary from [31, §4.4]. Then $(\hat{\tau}, \hat{T}, \leqslant)$ dominates $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$, and the change from $(\hat{\tau}, \hat{T}, \leqslant)$ to $(\tilde{\tau}, \tilde{T}, \leqslant)$ is locally finite. So Theorem 5.2 implies:

Corollary 5.8. *Let P be a projective $\mathbb{K}$-scheme and $\mathcal{A} = \mathrm{coh}(P)$, $K(\mathrm{coh}(P))$, $\mathfrak{F}_{\mathrm{coh}(P)}$ be as in* [29, Ex. 9.1 or 9.2]. *Suppose $(\tau, T, \leqslant)$ and $(\tilde{\tau}, \tilde{T}, \leqslant)$ are any two weak stability conditions on* $\mathrm{coh}(P)$ *from* [31, §4.4], *which may be defined using different ample line bundles $E, \tilde{E}$ on P. Then* (44)–(47) *hold for* $\mathrm{coh}(P)$, *as infinite convergent sums. If $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ are chosen from* [31, Ex.s 4.16 or 4.17], *so that they are permissible, then* (48)–(49) *also hold as infinite convergent sums.*

For *surfaces* changes between Gieseker stability conditions are globally finite.

Theorem 5.9. *Let P be a smooth projective surface, and $\mathcal{A} = \mathrm{coh}(P), K(\mathcal{A})$, $\mathfrak{F}_{\mathcal{A}}$ as in* [29, Ex. 9.1]. *Suppose $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ are any two permissible weak stability conditions on $\mathcal{A}$ from* [31, Ex.s 4.16–4.17], *which may be defined using different ample line bundles $E, \tilde{E}$. Then the changes between $(\tau, T, \leqslant)$ and $(\tilde{\tau}, \tilde{T}, \leqslant)$ are globally finite in the sense of Definition 5.1. So* (44)–(49) *hold as finite sums in* $\mathrm{CF}, \mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ *and* $\mathrm{CF}, \mathrm{SF}(\mathfrak{M}(K, \trianglelefteq, \mu)_{\mathcal{A}})$, *and* (53) *holds.*

The author can prove some partial results on global finiteness of changes between permissible weak stability conditions on $\mathcal{A} = \mathrm{coh}(P)$ when $\dim P \geqslant 3$, but we will not give them as they are complicated and inelegant. The main conclusion is this. Suppose $(\tau, T, \leqslant), (\tilde{\tau}, \tilde{T}, \leqslant)$ are any two permissible weak

stability conditions on $\mathcal{A}$ from [31, §4.4]. Then we can construct a finite sequence $(\tau,T,\leqslant)=(\tau_0,T_0,\leqslant),(\tau_1,T_1,\leqslant),\ldots,(\tau_n,T_n,\leqslant)=(\tilde\tau,\tilde T,\leqslant)$ of permissible weak stability conditions on $\mathcal{A}$, such that the changes between $(\tau_{i-1},T_{i-1},\leqslant)$ and $(\tau_i,T_i,\leqslant)$ are globally finite for $i=1,\ldots,n$. This implies that (53) holds, so $\mathcal{H}^{\rm pa}_\tau,\ldots,\bar{\mathcal{L}}^{\rm to}_\tau$ are independent of choice of $(\tau,T,\leqslant)$.

We show the *locally finite* condition in Theorem 5.2 is necessary.

Example 5.10. Take P to be the projective space $\mathbb{KP}^2$, with $\mathcal{A}=\mathrm{coh}(\mathbb{KP}^2)$ and $K(\mathcal{A}),\mathfrak{F}_{\mathcal{A}}$ as in [29, Ex. 9.1]. Define $(\tau,T,\leqslant)$ to be the trivial stability condition $(0,\{0\},\leqslant)$ on $\mathcal{A}$ and $(\tilde\tau,\tilde T,\leqslant)$ the Gieseker stability condition $(\gamma,G_2,\leqslant)$ of [31, Ex. 4.16], defined using $\mathcal{O}_{\mathbb{KP}^2}(1)$. Suppose $x_1,\ldots,x_n$ are distinct points in $\mathbb{KP}^2(\mathbb{K})$. Then there is an exact sequence

$$0\rightarrow Y_{x_1,\ldots,x_n}\rightarrow \mathcal{O}_{\mathbb{KP}^2}(-1)\oplus\mathcal{O}_{\mathbb{KP}^2}(0)\rightarrow \mathcal{O}_{\mathbb{KP}^2}(-1)\oplus\textstyle\bigoplus_{i=1}^n\mathcal{O}_{x_i}\rightarrow 0, \tag{54}$$

where $\mathcal{O}_{x_i}$ is the structure sheaf of x_i. Define $X=\mathcal{O}_{\mathbb{KP}^2}(-1)\oplus\mathcal{O}_{\mathbb{KP}^2}(0)$ in $\mathcal{A}$, $\alpha=[X]$ in $K(\mathcal{A})$, and $\kappa:\{1,2\}\rightarrow C(\mathcal{A})$ by $\kappa(1)=[Y_{x_1,\ldots,x_n}]=[\mathcal{O}_{\mathbb{KP}^2}(0)]-n[\mathcal{O}_{x_1}]$ and $\kappa(2)=[\mathcal{O}_{\mathbb{KP}^2}(-1)]+n[\mathcal{O}_{x_1}]$, so that $\kappa(\{1,2\})=\alpha$.

Calculation shows $X,Y_{x_1,\ldots,x_n}$ have Hilbert polynomials $p_X(t)=t^2+2t+1$ and $p_{Y_{x_1,\ldots,x_n}}=\frac{1}{2}t^2+\frac{3}{2}t+1-n$. Thus $\gamma\circ\kappa(1)=t^2+3t+2-2n$ and $\gamma\circ\kappa(2)=t^2+t+2n$, so that $\gamma\circ\kappa(1)>\gamma\circ\kappa(\{2\})$ and $S(\{1,2\},\leqslant,\kappa,0,\gamma)=-1$. As all sheaves are 0-semistable, (54) implies $\bar\delta_{\rm ss}^{\kappa(1)}(0)*\bar\delta_{\rm ss}^{\kappa(2)}(0)\neq 0$ over $[X]$.

Hence for each $n\geqslant 0$ we have distinct $(\{1,2\},\leqslant,\kappa)$ giving nonzero terms over $[X]$ in (45). Therefore (45) does not make sense, even with the notion of convergence in Definition 2.16. This works because the change from $(0,\{0\},\leqslant)$ to $(\gamma,G_2,\leqslant)$ is not locally finite.

The rest of the section proves Theorems 5.2, 5.4 and 5.9.

5.2 Transforming to a dominant weak stability condition

Let Assumption 3.5 hold and $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be weak stability conditions on $\mathcal{A}$ with $(\tilde\tau,\tilde T,\leqslant)$ *dominating* $(\tau,T,\leqslant)$ in the sense of Definition 3.16. Suppose $(\{1,\ldots,n\},\leqslant,\kappa)$ is $\mathcal{A}$-data with $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\neq 0$. Then Theorem 4.6 gives k,l with $\tau\circ\kappa(k)\leqslant\tau\circ\kappa(l)$ and $\tilde\tau\circ\kappa(k)\geqslant\tilde\tau\circ\kappa(l)$. But as $(\tilde\tau,\tilde T,\leqslant)$ dominates $(\tau,T,\leqslant)$, $\tau\circ\kappa(k)\leqslant\tau\circ\kappa(l)$ implies $\tilde\tau\circ\kappa(k)\leqslant\tilde\tau\circ\kappa(l)$, and so $\tilde\tau\circ\kappa(k)=\tilde\tau\circ\kappa(l)$. Therefore all $\tilde\tau\circ\kappa(i)$ are equal, by Theorem 4.6.

Using this and Definition 4.2 we see that if $(\tilde\tau,\tilde T,\leqslant)$ dominates $(\tau,T,\leqslant)$ and $(\{1,\ldots,n\},\leqslant,\kappa)$ is $\mathcal{A}$-data with $\kappa(\{1,\ldots,n\})=\alpha$ then

$$S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)=\begin{cases}1, & \tau\circ\kappa(1)>\cdots>\tau\circ\kappa(n),\ \tilde\tau\circ\kappa\equiv\tilde\tau(\alpha),\\ 0, & \text{otherwise.}\end{cases} \tag{55}$$

In our next theorem, (55) shows (56)–(57) are special cases of (44)–(45).

Theorem 5.11. *Let Assumption 3.5 hold, $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be weak stability conditions on $\mathcal{A}$ with $(\tilde\tau,\tilde T,\leqslant)$ dominating $(\tau,T,\leqslant)$, and $\alpha\in C(\mathcal{A})$. Then*

$$\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\ \tilde\tau\circ\kappa\equiv\tilde\tau(\alpha),\\ \tau\circ\kappa(1)>\cdots>\tau\circ\kappa(n),\ \kappa(\{1,\ldots,n\})=\alpha}} \delta_{\rm ss}^{\kappa(1)}(\tau)*\delta_{\rm ss}^{\kappa(2)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa(n)}(\tau)=\delta_{\rm ss}^{\alpha}(\tilde\tau), \tag{56}$$

$$\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\ \tilde\tau\circ\kappa\equiv\tilde\tau(\alpha),\\ \tau\circ\kappa(1)>\cdots>\tau\circ\kappa(n),\ \kappa(\{1,\ldots,n\})=\alpha}} \bar\delta_{\rm ss}^{\kappa(1)}(\tau)*\bar\delta_{\rm ss}^{\kappa(2)}(\tau)*\cdots*\bar\delta_{\rm ss}^{\kappa(n)}(\tau)=\bar\delta_{\rm ss}^{\alpha}(\tilde\tau). \tag{57}$$

These are potentially infinite sums, converging as in Definition 2.16.

Proof. Consider $\mathcal{A}$-data $(\{1,\ldots,n\},\leqslant,\kappa)$ with $\tau\circ\kappa(1)>\cdots>\tau\circ\kappa(n)$ and $\delta_{\rm ss}^{\kappa(1)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa(n)}(\tau)\neq 0$ at $[X]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$. By (19) $\mathrm{CF}^{\rm stk}(\boldsymbol\sigma(\{1,\ldots,n\}))\,\delta_{\rm ss}(\{1,\ldots,n\},\leqslant,\kappa,\tau)\neq 0$ at $[X]$, so we have $[(\sigma,\iota,\pi)]\in\mathcal{M}_{\rm ss}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}$ with $\sigma(\{1,\ldots,n\})=X$. From [29, Cor. 4.4], such $[(\sigma,\iota,\pi)]$ are in 1-1 correspondence with filtrations $0=A_0\subset\cdots\subset A_n=X$ with $S_i=A_i/A_{i-1}\cong\sigma(\{i\})$ for $i=1,\ldots,n$. Thus S_i is τ-semistable and $\tau([S_1])>\cdots>\tau([S_n])$. Theorem 3.13 now shows that $0=A_0\subset\cdots\subset A_n=X$ is the unique *Harder–Narasimhan filtration* of X. So n,κ and $[(\sigma,\iota,\pi)]$ are unique for fixed $[X]$.

Now $\mathrm{Iso}_{\mathbb{K}}\bigl([(\sigma,\iota,\pi)]\bigr)\cong\mathrm{Aut}((\sigma,\iota,\pi))\cong\mathrm{Aut}(A_0\subset\cdots\subset A_n=X)$ and $\mathrm{Iso}_{\mathbb{K}}\bigl([X]\bigr)\cong\mathrm{Aut}(X)$. But $\mathrm{Aut}(A_0\subset\cdots\subset A_n=X)=\mathrm{Aut}(X)$ as X determines $A_0\subset\cdots\subset A_n=X$ by Theorem 3.13. Thus $\boldsymbol\sigma(\{1,\ldots,n\})_*:\mathrm{Iso}_{\mathbb{K}}\bigl([(\sigma,\iota,\pi)]\bigr)\rightarrow\mathrm{Iso}_{\mathbb{K}}\bigl([X]\bigr)$ is an isomorphism of $\mathbb{K}$-groups, and $m_{\boldsymbol\sigma(\{1,\ldots,n\})}\bigl([(\sigma,\iota,\pi)]\bigr)=1$ in Definition 2.3. Equation (19) then gives $\delta_{\rm ss}^{\kappa(1)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa(n)}(\tau)=1$ at $[X]$.

As $(\tilde\tau,\tilde T,\leqslant)$ dominates $(\tau,T,\leqslant)$ we have $\tilde\tau\circ\kappa(1)\geqslant\cdots\geqslant\tilde\tau\circ\kappa(n)$, and the $\sigma(\{i\})$ are $\tilde\tau$-semistable by [31, Lem. 4.11]. It easily follows that X is $\tilde\tau$-semistable if and only if $\tilde\tau\circ\kappa\equiv\tilde\tau(\alpha)$. Thus, if X is $\tilde\tau$-semistable and $[X]=\alpha$ there exist unique n,κ in (56) with $\delta_{\rm ss}^{\kappa(1)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa(n)}(\tau)=1$ at $[X]$ and all other terms zero at $[X]$, and otherwise all terms in (56) are zero at $[X]$. That is, the l.h.s. of (56) is 1 at $[X]$ if $[X]\in\mathrm{Obj}_{\rm ss}^{\alpha}(\tilde\tau)$, and 0 otherwise. This proves (56).

Consider the map on $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ taking $[X]$ to (n,κ) constructed from the Harder–Narasimhan filtration as above. In the natural topology on $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$, it is easy to see n is an upper semicontinuous function of $[X]$, and κ is locally constant on the subset of $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ with fixed n. Thus the map $[X]\mapsto(n,\kappa)$ is *locally constructible*. Hence if $\mathfrak{G}\subset\mathfrak{Obj}_{\mathcal{A}}$ is a finite type substack then $[X]\mapsto(n,\kappa)$ takes only finitely many values on $\mathfrak{G}(\mathbb{K})$, and the restriction of (56) to $\mathfrak{G}$ has only finitely many nonzero terms, so (56) converges.

Next we extend all this to (57). The argument above shows that $[X]\in\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ lies in $\mathrm{Obj}_{\rm ss}^{\alpha}(\tilde\tau)$ if and only if $[X]=\boldsymbol\sigma(\{1,\ldots,n\})_*\bigl([(\sigma,\iota,\pi)]\bigr)$ for some $[(\sigma,\iota,\pi)]\in\mathcal{M}_{\rm ss}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}$ and n,κ as in (57), which are then unique. This gives a disjoint union

$$\mathrm{Obj}_{\rm ss}^{\alpha}(\tilde\tau)=\coprod_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\ \tilde\tau\circ\kappa\equiv\tilde\tau(\alpha),\\ \tau\circ\kappa(1)>\cdots>\tau\circ\kappa(n),\ \kappa(\{1,\ldots,n\})=\alpha}} \boldsymbol\sigma(\{1,\ldots,n\})_*\bigl(\mathcal{M}_{\rm ss}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}\bigr),$$

where for any finite type substack $\mathfrak{G} \subseteq \mathfrak{Obj}_{\mathcal{A}}$, only finitely many sets on the r.h.s. intersect $\mathfrak{G}(\mathbb{K})$. This translates into an identity in $\mathrm{L\dot{S}F}(\mathfrak{Obj}_{\mathcal{A}})$:

$$\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tilde{\tau}) = \sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\ \tilde{\tau}\circ\kappa\equiv\tilde{\tau}(\alpha),\\ \tau\circ\kappa(1)>\cdots>\tau\circ\kappa(n),\ \kappa(\{1,\ldots,n\})=\alpha}} \bar{\delta}_{\boldsymbol{\sigma}(\{1,\ldots,n\})_*(\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}})}, \tag{58}$$

which is a potentially infinite convergent sum.

Now for n, κ as in (57), the argument above shows that $\boldsymbol{\sigma}(\{1,\ldots,n\})_* : \mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}} \rightarrow \boldsymbol{\sigma}(\{1,\ldots,n\})_*\bigl(\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}\bigr)$ is a bijection, inducing isomorphisms of stabilizer groups. Here each side is the set of $\mathbb{K}$-points in a locally closed $\mathbb{K}$-substack of $\mathfrak{M}(\{1,\ldots,n\},\leqslant,\kappa)_{\mathcal{A}}$ and $\mathfrak{Obj}_{\mathcal{A}}$, and one can strengthen the argument to show that $\boldsymbol{\sigma}(\{1,\ldots,n\})$ induces a 1-isomorphism of these substacks. Therefore

$$\boldsymbol{\sigma}(\{1,\ldots,n\})_*\bigl(\bar{\delta}_{\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}}\bigr) = \bar{\delta}_{\boldsymbol{\sigma}(\{1,\ldots,n\})_*(\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}})}.$$

Combining this with Definition 3.14 and (19) gives

$$\bar{\delta}_{\mathrm{ss}}^{\kappa(1)}(\tau) * \cdots * \bar{\delta}_{\mathrm{ss}}^{\kappa(n)}(\tau) = \bar{\delta}_{\boldsymbol{\sigma}(\{1,\ldots,n\})_*(\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}})}. \tag{59}$$

Equation (57), and its convergence, then follow from (58) and (59). □

5.3 Inverting equations (56)–(57)

In the situation of §5.2, if $(\tilde{\tau},\tilde{T},\leqslant)$ dominates $(\tau,T,\leqslant)$ and $(\{1,\ldots,m\},\leqslant,\lambda)$ is $\mathcal{A}$-data with $\lambda(\{1,\ldots,m\})=\alpha$ then a similar argument to (55) shows that

$$S(\{1,\ldots,m\},\leqslant,\lambda,\tilde{\tau},\tau) = \begin{cases} (-1)^{m-1}, & \tau\circ\lambda(\{1,\ldots,i\})>\tau\circ\lambda(\{i+1,\ldots,m\}) \\ & \text{for } i=1,\ldots,m-1, \text{ and } \tilde{\tau}\circ\lambda\equiv\tilde{\tau}(\alpha), \\ 0, & \text{otherwise.} \end{cases} \tag{60}$$

Equation (60) shows (61)–(62) are special cases of (44)–(45).

Theorem 5.12. *Let Assumption 3.5 hold, $(\tau,T,\leqslant),(\tilde{\tau},\tilde{T},\leqslant)$ be weak stability conditions on $\mathcal{A}$ with $(\tilde{\tau},\tilde{T},\leqslant)$ dominating $(\tau,T,\leqslant)$ and the change from $(\tilde{\tau},\tilde{T},\leqslant)$ to $(\tau,T,\leqslant)$ locally finite, and $\alpha\in C(\mathcal{A})$. Then*

$$\sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,m\},\leqslant,\lambda):\ \tilde{\tau}\circ\lambda\equiv\tilde{\tau}(\alpha),\\ \tau\circ\lambda(\{1,\ldots,i\})>\tau\circ\lambda(\{i+1,\ldots,m\})\\ \text{for } 1\leqslant i<m,\ \lambda(\{1,\ldots,m\})=\alpha}} (-1)^{m-1}\delta_{\mathrm{ss}}^{\lambda(1)}(\tilde{\tau}) * \delta_{\mathrm{ss}}^{\lambda(2)}(\tilde{\tau}) * \cdots * \delta_{\mathrm{ss}}^{\lambda(m)}(\tilde{\tau}) = \delta_{\mathrm{ss}}^{\alpha}(\tau), \tag{61}$$

$$\sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,m\},\leqslant,\lambda):\ \tilde{\tau}\circ\lambda\equiv\tilde{\tau}(\alpha),\\ \tau\circ\lambda(\{1,\ldots,i\})>\tau\circ\lambda(\{i+1,\ldots,m\})\\ \text{for } 1\leqslant i<m,\ \lambda(\{1,\ldots,m\})=\alpha}} (-1)^{m-1}\bar{\delta}_{\mathrm{ss}}^{\lambda(1)}(\tilde{\tau}) * \bar{\delta}_{\mathrm{ss}}^{\lambda(2)}(\tilde{\tau}) * \cdots * \bar{\delta}_{\mathrm{ss}}^{\lambda(m)}(\tilde{\tau}) = \bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau). \tag{62}$$

These are potentially infinite sums, converging as in Definition 2.16.

Proof. As the change from $(\tilde\tau,\tilde T,\leqslant)$ to $(\tau,T,\leqslant)$ is locally finite, (60) implies that (61)–(62) converge. Equation (61) follows from

$$
\begin{aligned}
&\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,m\},\leqslant,\lambda):\ \tilde\tau\circ\lambda\equiv\tilde\tau(\alpha),\\ \tau\circ\lambda(\{1,\ldots,i\})>\tau\circ\lambda(\{i+1,\ldots,m\})\\ \text{for }1\leqslant i<m,\ \lambda(\{1,\ldots,m\})=\alpha}} (-1)^{m-1}\delta_{\rm ss}^{\lambda(1)}(\tilde\tau)*\cdots*\delta_{\rm ss}^{\lambda(m)}(\tilde\tau) = \sum_{\substack{\mathcal{A}\text{-data}\\ (\{1,\ldots,m\},\leqslant,\lambda):\\ \lambda(\{1,\ldots,m\})=\alpha}}\\
&S(\{1,\ldots,m\},\leqslant,\lambda,\tilde\tau,\tau)\cdot \sum_{\substack{\mathcal{A}\text{-data}\\ (\{1,\ldots,n_i\},\leqslant,\kappa_i),\\ i=1,\ldots,m:\\ \kappa_i(\{1,\ldots,n_i\})=\lambda_i,\\ \text{all } i=1,\ldots,m}} \begin{array}{l}\prod_{i=1}^m S(\{1,\ldots,n_i\},\leqslant,\kappa_i,\tau,\tilde\tau)\cdot\\ \bigl(\delta_{\rm ss}^{\kappa_1(1)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa_1(n_1)}(\tau)\bigr)*\cdots\\ *\bigl(\delta_{\rm ss}^{\kappa_m(1)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa_m(n_m)}(\tau)\bigr)=\end{array}\\
&\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \delta_{\rm ss}^{\kappa(1)}(\tau)*\cdots*\delta_{\rm ss}^{\kappa(n)}(\tau)\cdot\\
&\left[\sum_{m=1}^n \sum_{\substack{\psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}:\\ \psi\text{ is surjective,}\\ 1\leqslant i\leqslant j\leqslant n\text{ implies }\psi(i)\leqslant\psi(j),\\ \text{define }\lambda:\{1,\ldots,m\}\rightarrow C(\mathcal{A})\\ \text{by }\lambda(k)=\kappa(\psi^{-1}(k))}} \begin{array}{l} S(\{1,\ldots,m\},\leqslant,\lambda,\tilde\tau,\tau)\cdot\\ \displaystyle\prod_{k=1}^m S\bigl(\psi^{-1}(\{k\}),\leqslant,\kappa|_{\psi^{-1}(\{k\})},\tau,\tilde\tau\bigr)\end{array}\right]\\
&\qquad\qquad = \delta_{\rm ss}^\alpha(\tau).
\end{aligned}
\tag{63}
$$

Here in the first step we use (55) and (60) and substitute (56) in for $\delta_{\rm ss}^{\lambda(i)}(\tilde\tau)$, and in the second we replace the sums over m,λ and n_i,κ_i for $1\leqslant i\leqslant m$ by sums over n,κ and m,ψ,λ, where $n=n_1+\cdots+n_m$, and $\kappa:\{1,\ldots,n\}\rightarrow C(\mathcal{A})$ is given by $\kappa(j)=\kappa_i(k)$ if $j=m_1+\cdots+m_{i-1}+k$, $1\leqslant k\leqslant m_i$, and $\psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}$ is given by $\psi(j)=i$ if $m_1+\cdots+m_{i-1}<j\leqslant m_1+\cdots+m_i$, and use associativity of $*$. For the third and final step we note that $[\cdots]$ in the fourth line of (63) is $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tau)$ by (33), but this is 1 if $n=1$ and 0 otherwise by (32). So the only contribution is $n=1$ and $\kappa(1)=\alpha$, giving $\delta_{\rm ss}^\alpha(\tau)$.

As (63) involves infinite sums, we must also check our rearrangements of these are valid. Let $\mathfrak{G}\subseteq\mathfrak{Obj}_{\mathcal{A}}$ be a finite type $\mathbb{K}$-substack. We shall show that the restriction of (63) to $\mathfrak{G}$ has only finitely many nonzero terms at every stage, so our argument above just rearranges finite sums over $\mathfrak{G}$, and (63) holds as a convergent sum. Since the change from $(\tilde\tau,\tilde T,\leqslant)$ to $(\tau,T,\leqslant)$ is locally finite and $\mathfrak{G}(\mathbb{K})$ is constructible, by Definition 5.1 there are only finitely many $(\{1,\ldots,m\},\leqslant,\lambda)$ with $S(\{1,\ldots,m\},\leqslant,\lambda,\tilde\tau,\tau)\neq 0$ and

$$\mathfrak{G}(\mathbb{K})\cap\boldsymbol{\sigma}(\{1,\ldots,m\})_*\bigl(\mathcal{M}_{\rm ss}(\{1,\ldots,m\},\leqslant,\lambda,\tilde\tau)_{\mathcal{A}}\bigr)\neq\emptyset. \tag{64}$$

Thus there are finitely many nonzero terms over $\mathfrak{G}$ in the first step of (63).

The second and third steps of (63) are equivalent as they are a relabelling. Suppose n,κ,m,ψ,λ give a nonzero term over $\mathfrak{G}$ in the third step. Then $S(\{1,\ldots,m\},\leqslant,\lambda,\tilde\tau,\tau)\neq 0$ and $S(\psi^{-1}(\{k\}),\leqslant,\kappa|_{\psi^{-1}(\{k\})},\tau,\tilde\tau)\neq 0$ for $k=1,\ldots,m$, and

$$\mathfrak{G}(\mathbb{K})\cap\boldsymbol{\sigma}(\{1,\ldots,n\})_*\bigl(\mathcal{M}_{\rm ss}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}\bigr)\neq\emptyset. \tag{65}$$

Let $[(\sigma,\iota,\pi)]\in\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}$. Then $\sigma(\{i\})$ is τ-semistable for all i, so that $\sigma(\{i\})$ is $\tilde\tau$-semistable by [31, Lem. 4.11] as $(\tilde\tau,\tilde T,\leqslant)$ dominates $(\tau,T,\leqslant)$. Also, $S(\psi^{-1}(\{k\}),\leqslant,\kappa|_{\psi^{-1}(\{k\})},\tau,\tilde\tau)\neq 0$ implies that $\tilde\tau\circ\kappa$ is constant on $\psi^{-1}(\{k\})$ by (55). It follows that $\sigma(\psi^{-1}(\{k\}))$ is $\tilde\tau$-semistable for $k=1,\ldots,m$, which implies that $Q(\{1,\ldots,n\},\leqslant,\{1,\ldots,m\},\leqslant,\psi)_*$ maps $\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}\rightarrow\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,m\},\leqslant,\lambda,\tilde\tau)_{\mathcal{A}}$.

As $\boldsymbol\sigma(\{1,\ldots,n\})=\boldsymbol\sigma(\{1,\ldots,m\})\circ Q(\{1,\ldots,n\},\leqslant,\{1,\ldots,m\},\leqslant,\psi)$, this and (65) imply (64). Thus from the previous part, there are only finitely many possibilities for m,λ giving nonzero terms in the second and third steps of (63). Fix one such choice. Then any $(\{1,\ldots,n_i\},\leqslant,\kappa_i)$ for $i=1,\ldots,m$ giving a nonzero term in the second step comes from the (unique) τ-Harder–Narasimhan filtration of some point $[X_i]$ in

$$\boldsymbol\sigma(\{i\})_*\big[\boldsymbol\sigma(\{1,\ldots,m\})_*^{-1}(\mathfrak{G}(\mathbb{K}))\cap\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,m\},\leqslant,\lambda,\tilde\tau)_{\mathcal{A}}\big]. \qquad (66)$$

Since $\boldsymbol\sigma(\{1,\ldots,m\})$ here is of finite type by [29, Th. 8.4(b)] and $\mathfrak{G}(\mathbb{K})$ is constructible, $\boldsymbol\sigma(\{1,\ldots,m\})_*^{-1}(\mathfrak{G}(\mathbb{K}))$ is constructible, and so (66) is constructible. As in the proof of Theorem 5.2 the map $[X_i]\mapsto(n_i,\kappa_i)$ is locally constructible, and so takes only finitely many values on (66). Therefore there are only finitely many choices for n_i,κ_i giving nonzero terms, and the restriction of (63) to $\mathfrak{G}$ has only finitely many nonzero terms at each step. This proves (61). The proof of (62) is the same, substituting (57) in for $\bar\delta_{\mathrm{ss}}^{\lambda(i)}(\tilde\tau)$. $\square$

5.4 Proof of Theorem 5.2

Let $U\subseteq\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ be constructible, and suppose $(\{1,\ldots,n\},\leqslant,\kappa)$ is $\mathcal{A}$-data for which $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\neq 0$ and (43) holds. Then by (33) there exist $\mathcal{A}$-data $(\{1,\ldots,m\},\leqslant,\lambda)$ and surjective $\psi:\{1,\ldots,n\}\rightarrow\{1,\ldots,m\}$ with $i\leqslant j$ implies $\psi(i)\leqslant\psi(j)$ and $\lambda(k)=\kappa(\psi^{-1}(k))$ for $k=1,\ldots,m$, such that $S(\{1,\ldots,m\},\leqslant,\lambda,\hat\tau,\tilde\tau)\neq 0$ and $S\big(\psi^{-1}(\{k\}),\leqslant,\kappa|_{\psi^{-1}(\{k\})},\tau,\hat\tau\big)\neq 0$ for all k.

Since $(\hat\tau,\hat T,\leqslant)$ dominates $(\tau,T,\leqslant)$, the argument of Theorem 5.12 shows that $Q(\{1,\ldots,n\},\leqslant,\{1,\ldots,m\},\leqslant,\psi)_*$ maps $\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,n\},\leqslant,\kappa,\tau)_{\mathcal{A}}\rightarrow\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,m\},\leqslant,\lambda,\hat\tau)_{\mathcal{A}}$, and $U\cap\boldsymbol\sigma(\{1,\ldots,m\})_*\big(\mathcal{M}_{\mathrm{ss}}(\{1,\ldots,m\},\leqslant,\lambda,\hat\tau)_{\mathcal{A}}\big)\neq\emptyset$. As the change from $(\hat\tau,\hat T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ is locally finite, by Definition 5.1 there are only finitely many possibilities for m,λ. The argument of Theorem 5.12 then shows there are only finitely many possibilities for n,κ. Hence the change from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ is locally finite, as we want. This implies (44)–(45) are convergent in the sense of Definition 2.16.

To prove (44)–(45) we now substitute (56)–(57) with $\hat\tau,\lambda(i)$ in place of $\tilde\tau,\alpha$ for $i=1,\ldots,m$ into (61)–(62) with $\hat\tau,\tilde\tau$ in place of $\tilde\tau,\tau$ respectively. We then rearrange these equations using (33) exactly as for (63), but without using (32). The argument of Theorem 5.12 shows that for finite type $\mathfrak{G}\subseteq\mathfrak{Obj}_{\mathcal{A}}$ there are only finitely many nonzero terms over $\mathfrak{G}$ at each stage, so the rearrangements are valid and (44)–(45) hold.

Next we prove (46)–(47). Let $I,\preceq,\kappa,\phi$ be as in (46), and set $n_k=|\phi^{-1}(\{k\})|$ for $k\in K$. By Definition 4.3 $(\phi^{-1}(\{k\}),\preceq)$ is a total order, so there ex-

ists a unique bijection $\psi_k : \{1,\ldots,n_k\} \to \phi^{-1}(\{k\})$ with $\psi_k^*(\preceq) = \leqslant$. Since $\delta_{\rm ss}(K,\trianglelefteq,\mu,\tilde\tau) = \prod_{k\in K} \boldsymbol\sigma(\{k\})^*\bigl(\delta_{\rm ss}^{\mu(k)}(\tilde\tau)\bigr)$, taking the product over $k \in K$ of $\boldsymbol\sigma(\{k\})^*$ applied to (44) with $\mu(k)$ in place of α and using (19) gives

$$
\begin{aligned}
\delta_{\rm ss}(K,\trianglelefteq,\mu,\tilde\tau) &= \prod_{k\in K} \sum_{\substack{\mathcal{A}\text{-data}\\ (\{1,\ldots,n_k\},\leqslant,\kappa_k):\\ \kappa_k(\{1,\ldots,n_k\})=\mu(k)}} \begin{aligned}[t] &S(\{1,\ldots,n_k\},\leqslant,\kappa_k,\tau,\tilde\tau)\cdot \\ &\boldsymbol\sigma(\{k\})^*\circ \mathrm{CF}^{\rm stk}(\boldsymbol\sigma(\{1,\ldots,n_k\})) \\ &\delta_{\rm ss}(\{1,\ldots,n_k\},\leqslant,\kappa_k,\tau) \end{aligned} \\
&= \sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets } I}} \frac{1}{|I|!}\cdot \sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa)\text{ is }\mathcal{A}\text{-data,}\\ (I,\preceq,K,\phi)\text{ is dominant,}\\ \trianglelefteq=\mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k)\text{ for }k\in K}} \begin{aligned}[t] &\textstyle\prod_{k\in K} S(\phi^{-1}(\{k\}),\preceq,\kappa,\tau,\tilde\tau)\cdot \\ &\textstyle\prod_{k\in K}\bigl[\boldsymbol\sigma(\{k\})^*\circ \mathrm{CF}^{\rm stk}(\boldsymbol\sigma(\phi^{-1}(\{k\}))) \\ &\delta_{\rm ss}(\phi^{-1}(\{k\}),\preceq,\kappa,\tau)\bigr]. \end{aligned}
\end{aligned}
\tag{67}
$$

Here in the second line of (67) we note that as above, to each choice of $I,\preceq,\kappa,\phi$ in (46) we can associate n_k,ψ_k as above and $\kappa_k = \kappa\circ\psi_k$ as in the first line of (67). Conversely, the data n_k,κ_k for $k\in K$ determine $I,\preceq,\kappa,\phi$ uniquely up to isomorphism. However, the sum in (67) is not over isomorphism classes of quadruples $(I,\preceq,\kappa,\phi)$, but rather over isomorphism classes of I, followed by a sum over all $\preceq,\kappa,\phi$. It is easy to see that $(I,\preceq,\kappa,\phi)$ and $(I,\preceq',\kappa',\phi')$ are isomorphic if and only if $(\preceq',\kappa',\phi') = \pi_*(\preceq,\kappa,\phi)$ for some permutation $\pi : I\to I$, and different π give different $(\preceq',\kappa',\phi')$. So we include the factor $1/|I|!$ to cancel the number $|I|!$ of permutations of I.

As (67) involves infinite sums, we must also consider the convergence issues, and whether our rearrangements of sums are valid. Let $\mathfrak{G}\subseteq \mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}}$ be a finite type $\mathbb{K}$-substack. Then $\mathfrak{G}(\mathbb{K})$ is constructible, so $\boldsymbol\sigma(\{k\})_*(\mathfrak{G}(\mathbb{K}))$ is constructible in $\mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ for each $k\in K$, so by the convergence of (44)–(45) there are only finitely many choices of n_k,κ_k giving nonzero terms over $\mathfrak{G}$ in (67), and so only finitely many choices up to isomorphism of $I,\preceq,\kappa,\phi$. This proves (67), as a convergent infinite sum in $\mathrm{LCF}(\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}})$.

Now for $I,\preceq,\kappa,\phi$ as in (46), by applying the proof of [29, Th. 7.10] $|K|$ times, we can show that the following is a Cartesian square:

$$
\begin{tikzcd}
\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}} \arrow[r, "Q(I,\preceq,K,\trianglelefteq,\phi)"'] \arrow[d, "\prod_{k\in K} S(I,\preceq,\phi^{-1}(\{k\}))"] & \mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}} \arrow[d, "\prod_{k\in K}\boldsymbol\sigma(\{k\})"'] \\
\prod_{k\in K}\mathfrak{M}(\phi^{-1}(\{k\}),\preceq,\kappa)_{\mathcal{A}} \arrow[r, "\prod_{k\in K}\boldsymbol\sigma(\phi^{-1}(\{k\}))"] & \prod_{k\in K}\mathfrak{Obj}_{\mathcal{A}}^{\mu(k)} .
\end{tikzcd}
\tag{68}
$$

By [29, Th. 8.4] the rows of (68) are representable and the right 1-morphism is finite type, so the left 1-morphism is too as (68) is Cartesian. Applying Theorem 2.4 then shows that

$$
\begin{aligned}
&\textstyle\prod_{k\in K}\bigl[\boldsymbol\sigma(\{k\})^*\circ\mathrm{CF}^{\rm stk}\bigl(\boldsymbol\sigma(\phi^{-1}(\{k\}))\bigr)\delta_{\rm ss}(\phi^{-1}(\{k\}),\preceq,\kappa,\tau)\bigr] \\
&=\bigl(\textstyle\prod_{k\in K}\boldsymbol\sigma(\{k\})\bigr)^*\circ\mathrm{CF}^{\rm stk}\bigl(\prod_{k\in K}\boldsymbol\sigma(\phi^{-1}(\{k\}))\bigr)\bigl[\prod_{k\in K}\delta_{\rm ss}(\phi^{-1}(\{k\}),\preceq,\kappa,\tau)\bigr] \\
&=\mathrm{CF}^{\rm stk}\bigl(Q(I,\preceq,K,\trianglelefteq,\phi)\bigr)\circ\bigl(\prod_{k\in K}S(I,\preceq,\phi^{-1}(\{k\}))\bigr)^*\bigl[\prod_{k\in K}\delta_{\rm ss}(\phi^{-1}(\{k\}),\preceq,\kappa,\tau)\bigr] \\
&=\mathrm{CF}^{\rm stk}\bigl(Q(I,\preceq,K,\trianglelefteq,\phi)\bigr)\delta_{\rm ss}(I,\preceq,\kappa,\tau).
\end{aligned}
$$

Substituting this into the last line of (67) and using (30) then proves (46), and its convergence. For (47) we use the same argument with (45) instead of

(44), $(\cdots)_*$ instead of $\mathrm{CF}^{\mathrm{stk}}(\cdots)$, Theorem 2.8 instead of Theorem 2.4, and replacing products of functions such as $\prod_{k\in K}\boldsymbol{\sigma}(\{k\})^*\bigl(\delta_{\mathrm{ss}}^{\mu(k)}(\tilde\tau)\bigr)$ by expressions like $\bigl(\prod_{k\in K}\boldsymbol{\sigma}(\{k\})\bigr)^*\bigl(\bigotimes_{k\in K}\bar\delta_{\mathrm{ss}}^{\mu(k)}(\tilde\tau)\bigr)$.

To prove (48)–(49), let $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$ be permissible. Substituting (31) into the left hand side of (48) and rewriting gives

$$\sum_{\substack{\mathcal{A}\text{-data}\\(\{1,\ldots,m\},\leqslant,\lambda):\\ \lambda(\{1,\ldots,m\})=\alpha}}\;\sum_{1\leqslant l\leqslant m}\frac{(-1)^{l-1}}{l}\cdot\sum_{\substack{\text{surjective}\\ \xi:\{1,\ldots,m\}\to\{1,\ldots,l\}:\\ i\leqslant j\text{ implies }\xi(i)\leqslant\xi(j).\\ \text{Define }\mu:\{1,\ldots,l\}\to C(\mathcal{A})\text{ by}\\ \mu(a)=\lambda(\xi^{-1}(a)).\text{ Then }\tilde\tau\circ\mu\equiv\tilde\tau(\alpha)}}\begin{array}{l}\prod_{a=1}^{l}S(\xi^{-1}(\{a\}),\leqslant,\lambda,\tau,\tilde\tau)\cdot\\ \delta_{\mathrm{ss}}^{\lambda(1)}(\tau)*\cdots*\delta_{\mathrm{ss}}^{\lambda(m)}(\tau)\end{array}$$

$$=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,l\},\leqslant,\mu):\\ \mu(\{1,\ldots,l\})=\alpha,\ \tilde\tau\circ\mu\equiv\tilde\tau(\alpha)}}\frac{(-1)^{l-1}}{l}\cdot\delta_{\mathrm{ss}}^{\mu(1)}(\tilde\tau)*\cdots*\delta_{\mathrm{ss}}^{\mu(l)}(\tilde\tau)=\epsilon^{\alpha}(\tilde\tau),$$

using (22) at the first step, (44) at the second, and (20) at the third. These rearrangements are valid, and the equation holds as an infinite convergent sum, since (44) is convergent and (20), (22) have only finitely many nonzero terms. This proves (48) and its convergence, and (49) is the same, using (23), (45) and (21). The final part of Theorem 5.2 is immediate.

5.5 Proof of Theorem 5.4

Let Condition 4.1 hold for $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$. When $|I|=1$ equation (50) is trivial. Suppose by induction that $n>1$ and (50) holds whenever $|I|<n$, and let $\kappa:I\to C(\mathcal{A})$ with $|I|=n$. We shall prove (50) for I,κ.

As in [31, Ex. 7.10], define a quiver $Q=(Q_0,Q_1,b,e)$ to have vertices I and an edge $\overset{i}{\bullet}\to\overset{j}{\bullet}$ for all $i,j\in I$. Consider the abelian category nil-$\mathbb{C}Q$ of nilpotent $\mathbb{C}$-representations of Q, with data $K(\text{nil-}\mathbb{C}Q),\mathfrak{F}_{\text{nil-}\mathbb{C}Q}$ satisfying Assumption 3.5 as in [29, Ex. 10.6]. Then $K(\text{nil-}\mathbb{C}Q)=\mathbb{Z}^I$ and $C(\text{nil-}\mathbb{C}Q)=\mathbb{N}^I\setminus\{0\}$. Write elements of $\mathbb{Z}^I$ as $\gamma:I\to\mathbb{Z}$. For $i\in I$ define $e_i\in C(\text{nil-}\mathbb{C}Q)$ by $e_i(j)=1$ if $j=i$ and $e_i(j)=0$ otherwise. Then $\sum_{i\in I}e_i=1$ in $C(\text{nil-}\mathbb{C}Q)$. Define $\tau':C(\text{nil-}\mathbb{C}Q)\to T$ and $\tilde\tau':C(\text{nil-}\mathbb{C}Q)\to\tilde T$ by $\tau'(\gamma)=\tau\bigl(\sum_{i\in I}\gamma(i)\kappa(i)\bigr)$ and $\tilde\tau'(\gamma)=\tilde\tau\bigl(\sum_{i\in I}\gamma(i)\kappa(i)\bigr)$. Then Condition 4.1(iv),(v) imply $(\tau',T,\leqslant),(\tilde\tau',\tilde T,\leqslant)$ are weak stability conditions on nil-$\mathbb{C}Q$, which are permissible by [31, Cor. 4.13].

Apply (51) in nil-$\mathbb{C}Q$ with $\alpha=1\in C(\text{nil-}\mathbb{C}Q)$ and $\tau',\tilde\tau',J,\lambda$ in place of $\tau,\tilde\tau,I,\kappa$. By [31, Prop. 7.11(a)], if J is finite and $\lambda:J\to C(\text{nil-}\mathbb{C}Q)$ with $\lambda(J)=1$ then $|J|\leqslant|I|=n$, and if $|J|=n$ then there is a unique bijection $\imath:J\to I$ with $\lambda(j)=e_{\imath(j)}\in C(\mathcal{A})$ for all $j\in J$. The number of $\lambda:J\to C(\text{nil-}\mathbb{C}Q)$ is the number $|I|!$ of bijections $\imath$, which cancels the factor $1/|I|!$ in (51). Taking $J=I$ and $\lambda:i\mapsto e_i$, the definition of $\tau',\tilde\tau'$ implies that

$U(I,\preceq,\lambda,\tau',\tilde\tau')=U(I,\preceq,\kappa,\tau,\tilde\tau)$. Thus we may rewrite (51) as

$$\sum_{\substack{\text{total orders }\preceq\text{ on }I.\\ \text{Write }I=\{i_1,\ldots,i_n\},\\ i_1\preceq i_2\preceq\cdots\preceq i_n}} U(I,\preceq,\kappa,\tau,\tilde\tau)\cdot\epsilon^{e_{i_1}}(\tau')*\cdots*\epsilon^{e_{i_n}}(\tau')=\epsilon^1(\tilde\tau')- \tag{69}$$
$$\sum_{\substack{\text{iso classes of}\\ \text{finite sets }J\\ \text{with }|J|<n}}\frac{1}{|J|!}\sum_{\substack{\lambda:J\to C(\text{nil-}\mathbb{C}Q):\\ \lambda(J)=1}}\left[\sum_{\substack{\text{total orders }\preceq\text{ on }J.\\ \text{Write }J=\{j_1,\ldots,j_m\},\\ j_1\preceq j_2\preceq\cdots\preceq j_m}}\begin{array}{l}U(J,\preceq,\lambda,\tau',\tilde\tau')\cdot\\ \epsilon^{\lambda(j_1)}(\tau')*\cdots*\epsilon^{\lambda(j_m)}(\tau')\end{array}\right].$$

Now $\epsilon^1(\tilde\tau'),\epsilon^{\lambda(j_a)}(\tau')$ lie in $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\text{nil-}\mathbb{C}Q})$ by [31, Th. 7.8]. By induction, as $|J|<n$ the term $[\cdots]$ is a sum of commutators of $\epsilon^{\lambda(j_a)}(\tau')$ and so lies in $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\text{nil-}\mathbb{C}Q})$. So every term on the right of (69) lies in $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\text{nil-}\mathbb{C}Q})$, and thus so does the left hand side. However, [31, Prop. 7.11(c)] implies the subalgebra $A_{I,\tau'}$ of $\mathrm{CF}(\mathfrak{Obj}_{\text{nil-}\mathbb{C}Q})$ generated by the $\epsilon^{e_i}(\tau')$ for $i\in I$ is *freely generated*, so that $A_{I,\tau'}\cong\mathbb{Q}[I]$. We can then deduce from [30, Prop. 4.14] that the Lie algebra $A_{I,\tau'}\cap\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\text{nil-}\mathbb{C}Q})$ is freely generated by the $\epsilon^{e_i}(\tau')$ for $i\in I$, so that $A_{I,\tau'}\cap\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\text{nil-}\mathbb{C}Q})\cong\mathcal{L}[I]$. Equation (50) for I,κ follows, as the l.h.s. of (69) lies in $A_{I,\tau'}\cap\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\text{nil-}\mathbb{C}Q})$. This completes the proof.

5.6 Proof of Theorem 5.9

Let P be a smooth projective $\mathbb{K}$-scheme of dimension m and define $\mathcal{A}=\mathrm{coh}(P),K(\mathcal{A}),\mathfrak{F}_{\mathcal{A}}$ satisfying Assumption 3.5 as in [29, Ex. 9.1]. We allow m arbitrary for our first two propositions, and then restrict to $m=2$. As in [29, Ex. 9.1] the *Chern character* gives a natural injective group homomorphism

$$\mathrm{ch}:K(\mathrm{coh}(P))\to\textstyle\bigoplus_{i=1}^m H^{2i}(P,\mathbb{Z}). \tag{70}$$

Here $H^*(P,\mathbb{Z})$ makes sense when $\mathbb{K}=\mathbb{C}$. For general $\mathbb{K}$ we can instead use the *Chow ring* $A(P)$, but we will not worry about this. Write $\mathrm{ch}_i(\alpha)$ for the component of $\mathrm{ch}(\alpha)$ in $H^{2i}(P,\mathbb{Z})$. These may be written in terms of the Chern classes $c_i(X)$ and the rank $\mathrm{rk}(X)$ as on [22, p. 432] by

$$\mathrm{ch}_0([X])=\mathrm{rk}(X),\;\; \mathrm{ch}_1([X])=c_1(X),\;\; \mathrm{ch}_2([X])=\tfrac12\bigl(c_1(X)^2-2c_2(X)\bigr), \tag{71}$$

and so on. Now let E be an ample line bundle on P. Then by the Riemann–Roch Theorem [22, Th. A.4.1] the *Hilbert polynomial* p_X of X w.r.t. E is

$$p_X(n)=\chi(X\otimes E^n)=\int_P(1+c_1(E))^n\cdot\mathrm{ch}([X])\cdot\mathrm{td}(P). \tag{72}$$

Here $\int_P(\cdots)$ means the image of the component in $H^{2m}(P,\mathbb{Z})$ in $(\cdots)$ under the natural homomorphism $H^{2m}(P,\mathbb{Z})\to\mathbb{Z}$, and $\mathrm{td}(P)$ is the *Todd class* of P, given in terms of Chern classes as on [22, p. 432] by

$$\mathrm{td}(P)=1+\tfrac12\,c_1(P)+\tfrac1{12}\bigl(c_1(P)^2+c_2(P)\bigr)+\tfrac1{24}\,c_1(P)c_2(P)+\cdots. \tag{73}$$

For $0 \not\cong X \in \mathcal{A}$, write $\dim X$ for the dimension of the support of X in P. This depends only on the class $[X] \in C(\mathrm{coh}(P))$, and so defines a map $\dim : C(\mathrm{coh}(P)) \to \mathbb{N}$. Note that for all the weak stability conditions $(\tau, T, \leqslant)$ on $\mathcal{A}$ defined in [31, §4.4], $\tau(\alpha)$ is a polynomial of degree $\dim \alpha$ for all $\alpha \in C(\mathcal{A})$.

Proposition 5.13. *Let* $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$ *be weak stability conditions on* $\mathcal{A}$ *from* [31, §4.4] *and* $(\{1, \ldots, n\}, \leqslant, \kappa)$ *be* $\mathcal{A}$*-data with* $\kappa(\{1, \ldots, n\}) = \alpha \in C(\mathcal{A})$ *and* $S(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde\tau) \neq 0$*. Then* $\dim \circ \kappa \equiv \dim \alpha$*.*

Proof. Apply (33) with $(\hat\tau, \hat T, \leqslant) = (\delta, D_m, \leqslant)$ from [31, Ex. 4.18]. Then there exist m, ψ, λ in (33) with $S\big(\psi^{-1}(\{k\}), \leqslant, \kappa|_{\psi^{-1}(\{k\})}, \tau, \delta\big) \neq 0$ for $k = 1, \ldots, m$ and $S(\{1, \ldots, m\}, \leqslant, \lambda, \delta, \tilde\tau) \neq 0$. As $(\delta, D_m, \leqslant)$ dominates $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$, equations (55) and (60) then imply that $\delta \circ \lambda \equiv \delta(\alpha)$ and $\delta \circ \kappa|_{\psi^{-1}(\{k\})} \equiv \delta \circ \lambda(k)$ for $k = 1, \ldots, m$. Together these give $\delta \circ \kappa \equiv \delta(\alpha)$. But $\delta(\beta) = t^{\dim \beta}$ for $\beta \in C(\mathcal{A})$, so $\dim \circ \kappa \equiv \dim \alpha$. □

Here is a finiteness result for $\mathcal{A}$-data of dimension zero or one.

Proposition 5.14. *Suppose* $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$ *are weak stability conditions on* $\mathcal{A}$ *from* [31, Ex. 4.16 or 4.17]*. Let* $\alpha \in C(\mathcal{A})$ *with* $\dim \alpha = 0$ *or* 1*. Then there exist at most finitely many sets of* $\mathcal{A}$*-data* $(\{1, \ldots, n\}, \leqslant, \kappa)$ *with* $\kappa(\{1, \ldots, n\}) = \alpha$ *and* $S(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde\tau) \neq 0$*.*

Proof. If $\dim \alpha = 0$ and n, κ are as above then $\dim \circ \kappa \equiv 0$, so that $\tau \circ \kappa \equiv \tilde\tau \circ \kappa \equiv t^0$ in all the examples of [31, §4.4]. Thus the strict inequalities in Definition 4.2(a),(b) cannot hold, and $S(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde\tau) = 0$ unless $n = 1$ and $\kappa(1) = \alpha$. Therefore there is only one set of $\mathcal{A}$-data.

Let $\dim \alpha = 1$, and suppose for simplicity that P is *connected*. If it is not, we may apply the argument below to each connected component of P. Identify $H^{2m}(P, \mathbb{Z}) \cong \mathbb{Z}$. Let $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$ be defined using ample line bundles $E, \tilde E$ on P respectively. Then using (72)–(73) we find that

$$\tau(\alpha) = t + \frac{\mathrm{ch}_m(\alpha) + \frac{1}{2} c_1(P)\, \mathrm{ch}_{m-1}(\alpha)}{c_1(E)\, \mathrm{ch}_{m-1}(\alpha)}, \quad \tilde\tau(\alpha) = t + \frac{\mathrm{ch}_m(\alpha) + \frac{1}{2} c_1(P)\, \mathrm{ch}_{m-1}(\alpha)}{c_1(\tilde E)\, \mathrm{ch}_{m-1}(\alpha)}, \tag{74}$$

whichever of [31, Ex. 4.16] or [31, Ex. 4.17] are used to define $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$.

Suppose $(\{1, \ldots, n\}, \leqslant, \kappa)$ is $\mathcal{A}$-data with $S(\{1, \ldots, n\}, \leqslant, \kappa, \tau, \tilde\tau) \neq 0$ and $\kappa(\{1, \ldots, n\}) = \alpha$. Then $\dim \circ \kappa \equiv 1$ by Proposition 5.13, forcing $\mathrm{ch}_j(\kappa(i)) = 0$ for $j < m - 1$ and $c_1(E')\, \mathrm{ch}_{m-1}(\kappa(i)) > 0$ for all i and ample line bundles E' on P. Since $\sum_{i=1}^n \mathrm{ch}_{m-1}(\kappa(i)) = \mathrm{ch}_{m-1}(\alpha)$ it is not difficult to see there are only finitely many possible choices for n and $\mathrm{ch}_{m-1}(\kappa(i))$, $i = 1, \ldots, n$.

Let k, l be as in Theorem 4.6. Using (74) and $\tilde\tau \circ \kappa(k) \geqslant \tilde\tau(\alpha)$, $\tau \circ \kappa(k) \leqslant \tau \circ \kappa(i)$, $1 \leqslant c_1(\tilde E)\, \mathrm{ch}_{m-1}(\kappa(k)) \leqslant c_1(\tilde E)\, \mathrm{ch}_{m-1}(\alpha)$, $1 \leqslant c_1(E)\, \mathrm{ch}_{m-1}(\kappa(k))$ and $c_1(E)\, \mathrm{ch}_{m-1}(\kappa(i)) \leqslant c_1(E)\, \mathrm{ch}_{m-1}(\alpha)$ we deduce the first inequality of

$$\begin{aligned} c_1(E)\, \mathrm{ch}_{m-1}(\alpha) \cdot \min\big(\mathrm{ch}_m(\alpha) + \tfrac{1}{2} c_1(P)\, \mathrm{ch}_{m-1}(\alpha), 0\big) &\leqslant \\ \mathrm{ch}_m(\kappa(i)) + \tfrac{1}{2} c_1(P)\, \mathrm{ch}_{m-1}(\kappa(i)) &\leqslant \\ c_1(E)\, \mathrm{ch}_{m-1}(\alpha) \cdot \max\big(\mathrm{ch}_m(\alpha) + \tfrac{1}{2} c_1(P)\, \mathrm{ch}_{m-1}(\alpha), 0\big) & \end{aligned} \tag{75}$$

for all $i = 1, \dots, n$. The second follows in the same way from $\tilde\tau(\alpha) \geqslant \tilde\tau \circ \kappa(l)$ and $\tau \circ \kappa(i) \leqslant \tau \circ \kappa(l)$. For each of the finitely many choices for n and $\mathrm{ch}_{m-1}(\kappa(i))$, equation (75) shows there are only finitely many possibilities for $\mathrm{ch}_m(\kappa(i))$ in $H^{2m}(P,\mathbb{Z}) \cong \mathbb{Z}$. So there are finitely many choices for $\mathrm{ch}(\kappa(i))$, and thus for $\kappa(i)$ as (70) is injective. Hence there are only finitely many possibilities for n, κ. $\square$

Now suppose P is a smooth 2-dimensional $\mathbb{K}$-variety, that is, a $\mathbb{K}$-*surface*. Following [23, p. 71], define the *discriminant* of $X \in \mathrm{coh}(P)$ to be

$$\begin{aligned}\Delta(X) &= 2\,\mathrm{rk}(X) c_2(X) - \bigl(\mathrm{rk}(X) - 1\bigr) c_1(X)^2 \\ &= \mathrm{ch}_1([X])^2 - 2\,\mathrm{ch}_0([X])\,\mathrm{ch}_2([X])\end{aligned} \tag{76}$$

in $H^4(P,\mathbb{Z}) \cong \mathbb{Z}$, where the second line follows from (71). More generally, if $\alpha \in C(\mathrm{coh}(P))$ we write $\Delta(\alpha) = \mathrm{ch}_1(\alpha)^2 - 2\,\mathrm{ch}_0(\alpha)\,\mathrm{ch}_2(\alpha)$ in $\mathbb{Z}$. We are interested in discriminants because of the following important inequality due to Bogomolov [23, Th. 3.4.1] when $\mathrm{char}\,\mathbb{K} = 0$, and Langer [34, Th. 3.3] when $\mathrm{char}\,\mathbb{K} > 0$.

Theorem 5.15. *Let P be a $\mathbb{K}$-surface, $(\tau, T, \leqslant)$ a weak stability condition from* [31, Ex. 4.16 or 4.17] *on $\mathcal{A} = \mathrm{coh}(P)$ defined using an ample line bundle E, and $X \in \mathcal{A}$ be τ-semistable. Then $\Delta(X) \geqslant C\,\mathrm{rk}(X)^2(\mathrm{rk}(X) - 1)^2$, where $C = 0$ if $\mathrm{char}\,\mathbb{K} = 0$, and $C < 0$ depends only on $P, \mathrm{char}\,\mathbb{K}$ and E if $\mathrm{char}\,\mathbb{K} > 0$.*

Drawing on ideas from Huybrechts and Lehn [23, §4.C], we prove a finiteness result for dimension two $\mathcal{A}$-data $(\{1, \dots, n\}, \leqslant, \kappa)$. It is related to Yoshioka [46, §2], who proves that for P a smooth projective surface over $\mathbb{C}$ and $\alpha \in C(\mathrm{coh}(P))$ fixed the ample cone of P is divided into finitely many *chambers*, and if $(\gamma, G_2, \leqslant)$ is Gieseker stability w.r.t. an ample line bundle E then $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(\gamma)$ depends only on which chamber $c_1(E)$ lies in.

Theorem 5.16. *Let P be a smooth projective surface, and $\mathcal{A} = \mathrm{coh}(P), K(\mathcal{A})$ as in* [29, Ex. 9.1]. *Suppose $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$ are permissible weak stability conditions on $\mathcal{A}$ from* [31, Ex. 4.16 or 4.17]*, defined using ample line bundles $E, \tilde E$. Let $\alpha \in C(\mathcal{A})$ with $\dim \alpha = 2$. Then there exist at most finitely many sets of $\mathcal{A}$-data $(\{1, \dots, n\}, \leqslant, \kappa)$ with $\kappa(\{1, \dots, n\}) = \alpha$, $S(\{1, \dots, n\}, \leqslant, \kappa, \tau, \tilde\tau) \neq 0$, and $\Delta(\kappa(i)) \geqslant C\,\mathrm{rk}(\kappa(i))^2(\mathrm{rk}(\kappa(i)) - 1)^2$ for $i = 1, \dots, n$ and C as above, and if there are any such n, κ then $\Delta(\alpha) \geqslant C\,\mathrm{rk}(\alpha)^2(\mathrm{rk}(\alpha) - 1)^2$.*

Proof. Identifying $H^4(P,\mathbb{Z}) \cong \mathbb{Z}$, for $\beta \in C(\mathcal{A})$ with $\dim \beta = 2$ define

$$\mu(\beta) = c_1(\beta) c_1(E) / \mathrm{rk}(\beta) \quad \text{and} \quad \tilde\mu(\beta) = c_1(\beta) c_1(\tilde E) / \mathrm{rk}(\beta) \quad \text{in } \mathbb{Q}. \tag{77}$$

For $s \in [0,1]$ define $\mu_s(\beta) = (1 - s)\mu(\beta) + s\tilde\mu(\beta)$ in $\mathbb{R}$. By computing Hilbert polynomials w.r.t. $E, \tilde E$ using (71)–(73) and referring to [31, Ex.s 4.16–4.17], we can show that if $\beta, \gamma \in C(\mathcal{A})$ with $\dim \beta = 2 = \dim \gamma$ then $\mu(\beta) < \mu(\gamma)$ implies $\tau(\beta) < \tau(\gamma)$ and $\tilde\mu(\beta) < \tilde\mu(\gamma)$ implies $\tilde\tau(\beta) < \tilde\tau(\gamma)$. So we shall use $\mu, \tilde\mu$ as substitutes for $\tau, \tilde\tau$ in the proof, and μ_s for $s \in [0,1]$ interpolate between them.

Let n, κ be as in the theorem. Then $\dim \circ \kappa \equiv 2$ by Proposition 5.13. If $n = 1$ the only possibility is $\kappa(1) = \alpha$, so suppose $n > 1$. For $s \in [0,1]$ and $1 \leqslant i < n$, consider the three (not exhaustive) alternatives:

(a) $\mu_s\circ\kappa(\{1,\ldots,i\})>\mu_s(\alpha)$ and $\tau\circ\kappa(i)\leqslant\tau\circ\kappa(i+1)$;

(b) $\mu_s\circ\kappa(\{1,\ldots,i\})<\mu_s(\alpha)$ and $\tau\circ\kappa(i)>\tau\circ\kappa(i+1)$; or

(c) $\mu_s\circ\kappa(\{1,\ldots,i\})=\mu_s(\alpha)$.

The point of this is that as $\tilde\mu(\beta)<\tilde\mu(\gamma)$ implies $\tilde\tau(\beta)<\tilde\tau(\gamma)$ from above, when $s=1$ parts (a),(b) above imply Definition 4.2(a),(b), and Definition 4.2(a),(b) imply (a),(b) or (c). Thus, $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau)\neq 0$ and Definition 4.2 imply that one of (a)–(c) above hold for all $1\leqslant i<n$ when $s=1$.

Similarly, when $s=0$ parts (a)–(c) above are related in the same way to Definition 4.2(a),(b) with τ in place of $\tilde\tau$. But $S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tau)=0$ by (32) as $n>1$. Therefore Definition 4.2 implies that there exists $1\leqslant i<n$ for which neither (a) nor (b) above holds when $s=0$. As one of (a)–(c) holds for this i when $s=1$, the sign of $\mu_s\circ\kappa(\{1,\ldots,i\})-\mu_s(\alpha)$ must change over $[0,1]$, so there exist $s\in[0,1]$ and $1\leqslant i<n$ such that (c) holds. Choose such values s_0,i_0 of s,i with s_0 as large as possible. Then since one of (a)–(c) hold for all $1\leqslant i<n$ at $s=1$, it is easy to see that one of (a)–(c) hold for all $1\leqslant i<n$ at $s=s_0$, and (c) holds for $s=s_0$, $i=i_0$. Also $s_0\in\mathbb{Q}$, as $\mu,\tilde\mu$ take values in $\mathbb{Q}$.

Define $\alpha',\alpha''\in C(\mathcal{A})$ and $\mathcal{A}$-data $(\{1,\ldots,n'\},\leqslant,\kappa')$, $(\{1,\ldots,n''\},\leqslant,\kappa'')$ by $\alpha'=\kappa(\{1,\ldots,i_0\})$, $\alpha''=\kappa(\{i_0+1,\ldots,n\})$, $n'=i_0$, $n''=n-i_0$, $\kappa'(i)=\kappa(i)$, and $\kappa''(i)=\kappa(i+i_0)$. Then $\kappa'(\{1,\ldots,n'\})=\alpha'$, $\kappa''(\{1,\ldots,n''\})=\alpha''$. As one of (a)–(c) hold for all $1\leqslant i<n$ at $s=s_0$, and (c) holds for $s=s_0$, $i=i_0$, it is not difficult to see that the analogues of one of (a)–(c) for α' and $(\{1,\ldots,n'\},\leqslant,\kappa')$ hold for $s=s_0$ and all $1\leqslant i<n'$, and the analogues of one of (a)–(c) for α'' and $(\{1,\ldots,n''\},\leqslant,\kappa'')$ hold for $s=s_0$ and all $1\leqslant i<n''$.

Set $\xi=\mathrm{rk}(\alpha')c_1(\alpha'')-\mathrm{rk}(\alpha'')c_1(\alpha')$ in $H^2(P,\mathbb{Z})$. Then (71), (76) yield

$$\frac{1}{\mathrm{rk}(\alpha)}\,\Delta(\alpha)=\frac{1}{\mathrm{rk}(\alpha')}\,\Delta(\alpha')+\frac{1}{\mathrm{rk}(\alpha'')}\,\Delta(\alpha'')-\frac{\xi^2}{\mathrm{rk}(\alpha)\,\mathrm{rk}(\alpha')\,\mathrm{rk}(\alpha'')}\,. \tag{78}$$

As $s_0\in\mathbb{Q}\cap[0,1]$ we can write $s_0=p/(p+q)$ for integers $p,q\geqslant 0$ with $p+q>0$. Since (c) holds for $s=s_0$, $i=i_0$ we find that

$$\bigl(q\,c_1(E)+p\,c_1(\tilde E)\bigr)\xi=0. \tag{79}$$

But $E^q\otimes\tilde E^p$ is ample, so the *Hodge Index Theorem* [22, Th. V.1.9] implies that $\xi^2\leqslant 0$, and thus (78) gives

$$\frac{1}{\mathrm{rk}(\alpha)}\,\Delta(\alpha)\geqslant\frac{1}{\mathrm{rk}(\alpha')}\,\Delta(\alpha')+\frac{1}{\mathrm{rk}(\alpha'')}\,\Delta(\alpha''). \tag{80}$$

Suppose now that we know that

$$\Delta(\alpha')\geqslant C\,\mathrm{rk}(\alpha')^2(\mathrm{rk}(\alpha')-1)^2\quad\text{and}\quad\Delta(\alpha'')\geqslant C\,\mathrm{rk}(\alpha'')^2(\mathrm{rk}(\alpha'')-1)^2. \tag{81}$$

Then from (78) and $\mathrm{rk}(\alpha'),\mathrm{rk}(\alpha'')\leqslant\mathrm{rk}(\alpha)$ we see that

$$2C\,\mathrm{rk}(\alpha)^4(\mathrm{rk}(\alpha)-1)^2-\mathrm{rk}(\alpha)^2\Delta(\alpha)\leqslant\xi^2\leqslant 0,$$

so there are only finitely many possibilities for the integer ξ^2. Combining this with (79) we see as in the proof of [23, Lem. 4.C.2] that ξ lies in a bounded, and hence finite, subset of $H^2(P,\mathbb{Z})$.

Since $1 \leqslant \mathrm{rk}(\alpha'), \mathrm{rk}(\alpha'') \leqslant \mathrm{rk}(\alpha)$ there are only finitely many choices for $\mathrm{ch}_0(\alpha') = \mathrm{rk}(\alpha')$ and $\mathrm{ch}_0(\alpha'') = \mathrm{rk}(\alpha'')$. But as $c_1(\alpha') + c_1(\alpha'') = c_1(\alpha)$, $\mathrm{rk}(\alpha'), \mathrm{rk}(\alpha'')$ and ξ determine $\mathrm{ch}_1(\alpha') = c_1(\alpha')$ and $\mathrm{ch}_1(\alpha'') = c_1(\alpha'')$, so there are only finitely many possibilities for these. From (76) and (81) we see that

$$\begin{aligned}
\mathrm{ch}_2(\alpha') &\leqslant \mathrm{ch}_1(\alpha')^2/2\,\mathrm{ch}_0(\alpha') - \tfrac{1}{2}C\,\mathrm{ch}_0(\alpha')\bigl(\mathrm{ch}_0(\alpha')-1\bigr)^2,\\
\mathrm{ch}_2(\alpha'') &\leqslant \mathrm{ch}_1(\alpha'')^2/2\,\mathrm{ch}_0(\alpha'') - \tfrac{1}{2}C\,\mathrm{ch}_0(\alpha'')\bigl(\mathrm{ch}_0(\alpha'')-1\bigr)^2,
\end{aligned}$$

so as $\mathrm{ch}_2(\alpha') + \mathrm{ch}_2(\alpha'') = \mathrm{ch}_2(\alpha)$ we see that once $\mathrm{ch}_i(\alpha'), \mathrm{ch}_i(\alpha'')$ are fixed for $i = 0, 1$ there are only finitely many choices for $\mathrm{ch}_2(\alpha'), \mathrm{ch}_2(\alpha'')$ in $\mathbb{Z}$. Thus, given α, C there are only finitely many possibilities for $\mathrm{ch}(\alpha'), \mathrm{ch}(\alpha'')$, and hence for α', α'' as (70) is injective.

We can now prove the following inductive hypothesis, for given $l \geqslant 1$:

$(*_l)$ Suppose $\alpha \in C(\mathcal{A})$ with $\dim\alpha = 2$ and $\mathrm{rk}(\alpha) \leqslant l$, and $(\{1,\ldots,n\}, \leqslant, \kappa)$ is $\mathcal{A}$-data with $\kappa(\{1,\ldots,n\}) = \alpha$ and $\Delta(\kappa(i)) \geqslant C\,\mathrm{rk}(\kappa(i))^2(\mathrm{rk}(\kappa(i))-1)^2$ for $i = 1,\ldots,n$, such that for some $s \in [0,1]$ and all $1 \leqslant i < n$ one of (a)–(c) above holds. Then $\Delta(\alpha) \geqslant C\,\mathrm{rk}(\alpha)^2(\mathrm{rk}(\alpha)-1)^2$. Moreover, for fixed α there are only finitely many possibilities for such n, κ.

When $l = 1$ this is trivial as the only possibility is $n = 1$ and $\kappa(1) = \alpha$. Suppose by induction that $(*_l)$ holds for $l \geqslant 1$, and let $\alpha \in C(\mathcal{A})$ with $\mathrm{rk}(\alpha) = l+1$, and n, κ be as in $(*_{l+1})$. Construct $s_0 \in [0, s]$ and α', n', κ', α'', n'', κ'' as in the first part of the proof. Then $\mathrm{rk}(\alpha'), \mathrm{rk}(\alpha'') \leqslant l$, so α', n', κ' and α'', n'', κ'' satisfy the conditions of $(*_l)$, which implies (81) holds. Combining this with (80) and $\mathrm{rk}(\alpha) = \mathrm{rk}(\alpha') + \mathrm{rk}(\alpha'')$ gives $\Delta(\alpha) \geqslant C\,\mathrm{rk}(\alpha)^2(\mathrm{rk}(\alpha)-1)^2$, as we want.

By the first part there are only finitely many possibilities for α', α''. For each of these, $(*_l)$ shows there are only finitely many choices for n', κ' and n'', κ''. But these determine n, κ, so there are only finitely many possibilities for n, κ, completing the inductive step. Thus $(*_l)$ holds for all $l \geqslant 1$. The theorem now follows from the first part of the proof, which showed that for n, κ as in the theorem the hypotheses of $(*_l)$ hold with $s = 1$. □

Definition 5.1, Proposition 5.14 and Theorems 5.15–5.16 show the change from $(\tau, T, \leqslant)$ to $(\tilde\tau, \tilde T, \leqslant)$ is *globally finite*, so the change from $(\tilde\tau, \tilde T, \leqslant)$ to $(\tau, T, \leqslant)$ is too by symmetry. Theorem 5.9 now follows from Corollary 5.8.

6 Invariants

Given a permissible weak stability condition $(\tau, T, \leqslant)$ on $\mathcal{A}$, we now consider how best to define *systems of invariants* $I_{\mathrm{ss}}(\cdots)$ of $\mathcal{A}$ and $(\tau, T, \leqslant)$ which 'count' τ-semistable objects in class $\alpha \in K(\mathcal{A})$, or more generally 'count' $(I, \preceq, \kappa)$-configurations (σ, ι, π) with $\sigma(\{i\})$ τ-semistable for all $i \in I$. Obviously there are

many ways of doing this, so we need to decide what are the most interesting, or useful, ways to define invariants. For instance, we could ask that the invariants we choose can be calculated in examples, or are important in other areas of mathematics.

The criterion we shall use to select interesting invariants is that *they should satisfy natural identities*, and the more identities the better. Of course, this is not unrelated to other criteria; for instance, such identities are powerful tools for calculating the invariants in examples, as we shall see, and may be a reason for the invariants to be important in other areas.

We shall divide the identities we are interested in into *additive identities*, for which the $I_{\rm ss}(\cdots)$ should take values in an abelian group or $\mathbb{Q}$-vector space, and *multiplicative identities*, for which the $I_{\rm ss}(\cdots)$ should take values in a ring or $\mathbb{Q}$-algebra. Here is a very general way of defining invariants satisfying useful additive identities.

Definition 6.1. Let $\mathbb{K},\mathcal{A},K(\mathcal{A})$ satisfy Assumption 3.5, and suppose Λ is a $\mathbb{Q}$-vector space and $\rho:\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\Lambda$ a $\mathbb{Q}$-linear map. Let $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. For all $\mathcal{A}$-data $(I,\preceq,\kappa)$, define *invariants* $I_{\rm ss}(I,\preceq,\kappa,\tau)$ in Λ by $I_{\rm ss}(I,\preceq,\kappa,\tau)=\rho\circ\boldsymbol\sigma(I)_*\,\bar\delta_{\rm ss}(I,\preceq,\kappa,\tau)$.

Remark 6.2. Here are some ways we might choose the linear map ρ:

(a) When $\mathrm{char}\,\mathbb{K}=0$ we could take $\rho=\rho'\circ\pi^{\rm stk}_{\mathfrak{Obj}_{\mathcal{A}}}$, for some linear $\rho':\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\Lambda$. This would have the advantage of making the meaning of the invariants – what they are 'counting' – clearer. For example, $I^\alpha_{\rm st}(\tau)$ in (83) below would measure the moduli space $\mathrm{Obj}^\alpha_{\rm st}(\tau)$ of τ-stable elements in class α, rather than the more obscure stack function $\bar\delta^\alpha_{\rm st}(\tau)$ of [31, §8]. In the same way, we could take $\rho=\rho'\circ\bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}$ for some linear $\rho':\bar{\mathrm{SF}}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}},\Theta,\Omega)\rightarrow\Lambda$. Then as in the discussion before [31, Th. 8.7], we could interpret $I^\alpha_{\rm st}(\tau)$ as counting 'virtual τ-stables' in class α, etc.

(b) As in §6.1 we could take $\rho=\rho'\circ\Pi_*$, where $\Pi:\mathfrak{Obj}_{\mathcal{A}}\rightarrow\mathrm{Spec}\,\mathbb{K}$ is the projection, Π_* maps $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K})$ (not $\mathrm{SF}(\mathrm{Spec}\,\mathbb{K})$ as Π is not representable), and $\rho':\underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K})\rightarrow\Lambda$ is linear. Then $I_{\rm ss}(I,\preceq,\kappa,\tau)=\rho'([\mathcal{M}_{\rm ss}(I,\preceq,\kappa,\tau)])$, so $I_{\rm ss}(I,\preceq,\kappa,\tau)$ depends only on the moduli space $\mathcal{M}_{\rm ss}(I,\preceq,\kappa,\tau)$ and not on its projection to $\mathfrak{Obj}_{\mathcal{A}}$.

(c) We could take Λ to be a $\mathbb{Q}$-algebra and $\rho:\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\Lambda$ to be an *algebra morphism*, such as those constructed in [30, §6]. This would imply *multiplicative identities* on the invariants in Λ. In the same way, we could restrict to invariants coming from $\bar{\mathcal{L}}^{\rm pa}_\tau$ and take $\rho:\mathrm{SF}^{\rm ind}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\Lambda$ to be a *Lie algebra morphism*, as in [30, §6.6].

Applying $\rho\circ\boldsymbol\sigma(K)_*$ to (47) in Theorem 5.2 gives:

Theorem 6.3. *Let Assumption 3.5 hold, $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be permissible weak stability conditions on $\mathcal{A}$, and $I_{\rm ss}(*)$ be as in Definition 6.1. Suppose the change from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ is globally finite, and there exists a weak stability*

condition $(\hat\tau,\hat T,\leqslant)$ *on* $\mathcal{A}$ *dominating* $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ *with the change from* $(\hat\tau,\hat T,\leqslant)$ *to* $(\tilde\tau,\tilde T,\leqslant)$ *locally finite. Then for all* $\mathcal{A}$*-data* $(K,\trianglelefteq,\mu)$ *the following holds in* Λ*, with only finitely many nonzero terms:*

$$\sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets } I}}\frac{1}{|I|!}\cdot\sum_{\substack{\preceq,\kappa,\phi:\ (I,\preceq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ (I,\preceq,K,\phi)\text{ is dominant},\\ \trianglelefteq=\mathcal{P}(I,\preceq,K,\phi),\\ \kappa(\phi^{-1}(k))=\mu(k)\text{ for }k\in K}}\begin{aligned}&T(I,\preceq,\kappa,K,\phi,\tau,\tilde\tau)\cdot\\ &I_{\rm ss}(I,\preceq,\kappa,\tau)=\\ &I_{\rm ss}(K,\trianglelefteq,\mu,\tilde\tau).\end{aligned}\tag{82}$$

Knowing $I_{\rm ss}(I,\preceq,\kappa,\tau)$ for all $(I,\preceq,\kappa)$ is equivalent to knowing the restriction $\rho:\bar{\mathcal{H}}^{\rm pa}_\tau\rightarrow\Lambda$. We know from §5 that under mild conditions, if $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$ are permissible weak stability conditions on $\mathcal{A}$ then $\bar{\mathcal{H}}^{\rm pa}_\tau=\bar{\mathcal{H}}^{\rm pa}_{\tilde\tau}$, as in (53), so knowing $I_{\rm ss}(I,\preceq,\kappa,\tau)$ for all $(I,\preceq,\kappa)$ is equivalent to knowing $I_{\rm ss}(I,\preceq,\kappa,\tilde\tau)$ for all $(I,\preceq,\kappa)$. Theorem 6.3 makes this equivalence explicit.

In the same way, subsystems of the $I_{\rm ss}(*,\tau)$ are equivalent to knowing the restrictions of ρ to $\bar{\mathcal{H}}^{\rm to}_\tau,\bar{\mathcal{L}}^{\rm pa}_\tau$ and $\bar{\mathcal{L}}^{\rm to}_\tau$. For example, knowing $I_{\rm ss}(I,\preceq,\kappa,\tau)$ for all $(I,\preceq,\kappa)$ with $\preceq$ a *total order* is equivalent to knowing $\rho:\bar{\mathcal{H}}^{\rm to}_\tau\rightarrow\Lambda$. If $(K,\trianglelefteq)$ is a total order then all $(I,\preceq)$ occurring in (82) are total orders, corresponding to the fact that $\bar{\mathcal{H}}^{\rm to}_\tau=\bar{\mathcal{H}}^{\rm to}_{\tilde\tau}$. Thus, the $I_{\rm ss}(I,\preceq,\kappa,\tau)$ with $(I,\preceq)$ a total order form a *closed subsystem* of the system $I_{\rm ss}(*,\tau)$ under change of stability condition.

We can also define other families of invariants $I_{\rm si},I_{\rm st},I_{\rm ss}^{\rm b},I_{\rm si}^{\rm b},I_{\rm st}^{\rm b}(I,\preceq,\kappa,\tau)$ in Λ by applying $\rho\circ\boldsymbol\sigma(I)_*$ to the stack functions $\bar\delta_{\rm si},\bar\delta_{\rm st},\bar\delta_{\rm ss}^{\rm b},\bar\delta_{\rm si}^{\rm b},\bar\delta_{\rm st}^{\rm b}(I,\preceq,\kappa,\tau)$ of [31, §8], and $I^\alpha_{\rm ss},I^\alpha_{\rm si},I^\alpha_{\rm st},J^\alpha(\tau)$ by applying ρ to $\bar\delta^\alpha_{\rm ss},\bar\delta^\alpha_{\rm si},\bar\delta^\alpha_{\rm st},\bar\epsilon^\alpha(\tau)$. Then the identities of [31, §8] yield many additive identities between these families of invariants. For instance, the stack function analogue of (14) implies that

$$I^\alpha_{\rm st}(\tau)=\sum_{\substack{\text{iso. classes}\\ \text{of finite sets }I}}\frac{1}{|I|!}\cdot\sum_{\substack{\preceq,\kappa:\ (I,\preceq,\kappa)\text{ is }\mathcal{A}\text{-data},\\ \kappa(I)=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}I_{\rm ss}(I,\preceq,\kappa,\tau)\cdot\sum_{\text{p.o.s }\lesssim\text{ on }I\text{ dominating }\preceq}n(I,\preceq,\lesssim)\,N(I,\lesssim).\tag{83}$$

Here $I^\alpha_{\rm st}(\tau)$ is an invariant which 'counts' τ-stable objects in class $\alpha\in K(\mathcal{A})$ (though see Remark 6.2(a) on this point). One overall moral is that all the invariants can be written in terms of the $I_{\rm ss}(*,\tau)$, so we use $I_{\rm ss}(*,\tau)$ for preference. The other invariants will be useful for some purposes though; for instance, from [31, Def. 8.9] we see that $\rho:\bar{\mathcal{L}}^{\rm pa}_\tau\rightarrow\Lambda$ is determined by the $I_{\rm si}^{\rm b}(I,\preceq,\kappa,\tau)$ or $I_{\rm st}^{\rm b}(I,\preceq,\kappa,\tau)$ for *connected* $(I,\preceq)$, so these $I_{\rm si}^{\rm b}(*,\tau)$ or $I_{\rm st}^{\rm b}(*,\tau)$ form a closed subsystem under change of stability condition.

In the rest of the section we study examples in which Λ is a $\mathbb{Q}$-algebra and the invariants satisfy additional multiplicative identities. The obvious way to do this is to arrange that $\rho:\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\Lambda$ is an algebra morphism, or $\rho:\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\Lambda$ a Lie algebra morphism, as in the constructions of [30, §6]. But in [30, §5.1] we also define multilinear operations $P_{(I,\preceq)}$ on $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$, and in some cases we can arrange for these to commute with operations $P_{(I,\preceq)}$ on Λ.

Having multiplicative identities means that the full system of invariants $I_{\rm ss}(I,\preceq,\kappa,\tau)$ is wholly determined by a smaller generating subset of invariants.

For instance, if ρ is an algebra morphism then ρ on $\bar{\mathcal{H}}_\tau^{\rm pa}$ or $\bar{\mathcal{H}}_\tau^{\rm to}$ is determined by its value on a set of generators for $\bar{\mathcal{H}}_\tau^{\rm pa}$ or $\bar{\mathcal{H}}_\tau^{\rm to}$, such as the $\bar\delta_{\rm ss}^\alpha(\tau)$ or $\bar\epsilon^\alpha(\tau)$ for $\bar{\mathcal{H}}_\tau^{\rm to}$. This reduces the amount of data needed to specify the $I_{\rm ss}(*,\tau)$, and so simplifies the problem of computing invariants in examples. So where we can we will focus on the invariants $I_{\rm ss}^\alpha(\tau), J^\alpha(\tau)$ associated to $\bar\delta_{\rm ss}^\alpha(\tau),\bar\epsilon^\alpha(\tau)$.

6.1 Multiplicative relations from disconnected $(I,\preceq)$

As in [31, §7.2], if $(I,\preceq)$ is a finite poset let $\approx$ be the equivalence relation on I generated by $i\approx j$ if $i\preceq j$ or $j\preceq i$, and define the *connected components* of $(I,\preceq)$ to be the $\approx$-equivalence classes. We shall now study invariants $I_{\rm ss}(I,\preceq,\kappa,\tau)$ that are multiplicative over disjoint unions of connected components.

Definition 6.4. Let Assumption 3.5 hold and $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. Suppose Λ is a commutative $\mathbb{Q}$-algebra and ρ' : $\underline{\rm SF}(\operatorname{Spec}\mathbb{K})\rightarrow\Lambda$ an algebra morphism. As in Remark 6.2(b), define invariants $I_{\rm ss}(I,\preceq,\kappa,\tau)$ in Λ for all $\mathcal{A}$-data $(I,\preceq,\kappa)$ by

$$I_{\rm ss}(I,\preceq,\kappa,\tau)=\rho'\circ\Pi_*\,\bar\delta_{\rm ss}(I,\preceq,\kappa,\tau)=\rho'\bigl([\mathcal{M}_{\rm ss}(I,\preceq,\kappa,\tau)]\bigr), \qquad (84)$$

where $\Pi:\mathfrak{Obj}_{\mathcal{A}}\rightarrow\operatorname{Spec}\mathbb{K}$ is the projection and $\Pi_*:{\rm SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow\underline{\rm SF}(\operatorname{Spec}\mathbb{K})$.

There are many ways of constructing such ρ'. For example, we can take $\Lambda=\underline{\rm SF}(\operatorname{Spec}\mathbb{K})$ and $\rho'={\rm id}_\Lambda$. Or for any Υ,Λ satisfying Assumption 2.10, such as Example 2.11 or others in [28, §4.1], we can take ρ' to be the morphism Υ' of Theorem 2.13. Also, using the notation of [27, §4.4], if $\operatorname{char}\mathbb{K}=0$ and w is any *allowable multiplicative weight function* on affine algebraic $\mathbb{K}$-groups taking values in $\mathbb{Q}$, then $\Lambda=\mathbb{Q}$ and $\rho':[\mathfrak{R}]\mapsto\chi_w(\mathfrak{R}(\mathbb{K}))$ defines an algebra morphism. Examples of such w are given in [27, Prop. 4.16].

These invariants satisfy a multiplicative identity. Note that (85) holds without any additional conditions on $\mathcal{A}$, in contrast to the identities of §6.2–§6.5, which require assumptions on the groups $\operatorname{Ext}^i(X,Y)$ for $X,Y\in\mathcal{A}$.

Proposition 6.5. *Let Assumption 3.5 hold and $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$, and define invariants $I_{\rm ss}(I,\preceq,\kappa,\tau)$ as in Definition 6.4. Suppose $(J,\lesssim,\lambda)$, $(K,\trianglelefteq,\mu)$ are $\mathcal{A}$-data with $J\cap K=\emptyset$. Define $\mathcal{A}$-data $(I,\preceq,\kappa)$ by $I=J\amalg K$, $\kappa|_J=\lambda$, $\kappa|_K=\mu$, and $i\preceq i'$ for $i,i'\in I$ if either $i,i'\in J$ and $i\lesssim i'$, or $i,i'\in K$ and $i\trianglelefteq i'$. Then*

$$I_{\rm ss}(I,\preceq,\kappa,\tau)=I_{\rm ss}(J,\lesssim,\lambda,\tau)\cdot I_{\rm ss}(K,\trianglelefteq,\mu,\tau). \qquad (85)$$

Equation (85) *also holds with $I_{\rm ss}$ replaced by $I_{\rm si},I_{\rm st},I_{\rm ss}^{\rm b},I_{\rm si}^{\rm b}$ or $I_{\rm st}^{\rm b}$ throughout.*

Proof. As in [29, §7.4] we can prove we have a 1-isomorphism:

$$S(I,\preceq,J)\times S(I,\preceq,K):\mathfrak{M}(I,\preceq,\kappa)_{\mathcal{A}}\longrightarrow\mathfrak{M}(J,\lesssim,\lambda)_{\mathcal{A}}\times\mathfrak{M}(K,\trianglelefteq,\mu)_{\mathcal{A}}.$$

This implies $\bigl(S(I,\preceq,J)\times S(I,\preceq,K)\bigr)_*\bigl(\mathcal{M}_{\rm ss}(I,\preceq,\kappa,\tau)_{\mathcal{A}}\bigr)=\mathcal{M}_{\rm ss}(J,\lesssim,\lambda,\tau)_{\mathcal{A}}\times\mathcal{M}_{\rm ss}(K,\trianglelefteq,\mu,\tau)_{\mathcal{A}}$, so $[\mathcal{M}_{\rm ss}(I,\preceq,\kappa,\tau)_{\mathcal{A}}]=[\mathcal{M}_{\rm ss}(J,\lesssim,\lambda,\tau)_{\mathcal{A}}]\cdot[\mathcal{M}_{\rm ss}(K,\trianglelefteq,\mu,\tau)_{\mathcal{A}}]$

in the algebra $\underline{\rm SF}(\operatorname{Spec}\mathbb{K})$. Equation (85) then follows from (84) and ρ' an algebra morphism. The analogues for $I_{\rm si},\ldots,I_{\rm st}^{\rm b}$ can be deduced from (85) and the additive identities relating $I_{\rm ss}(*,\tau)$ and $I_{\rm si},\ldots,I_{\rm st}^{\rm b}(*,\tau)$ that follow from the additive identities on $\bar\delta_{\rm ss},\ldots,\bar\delta_{\rm st}^{\rm b}(*,\tau)$ in [31, §8]. $\square$

We see from (85) that all $I_{\rm ss}(I,\preceq,\kappa,\tau)$ are determined by the subset of $I_{\rm ss}(I,\preceq,\kappa,\tau)$ with $(I,\preceq)$ *connected*. Equivalently, they are determined by the subset of $I_{\rm si}^{\rm b}(I,\preceq,\kappa,\tau)$ with $(I,\preceq)$ connected. But by [31, Def. 8.9], the Lie algebra $\bar{\mathcal L}_\tau^{\rm pa}$ is spanned by $\bar\delta_{\rm si}^{\rm b}(I,\preceq,\kappa,\tau)$ with $(I,\preceq)$ connected. Therefore (85) implies that $\rho=\rho'\circ\Pi_*:\bar{\mathcal H}_\tau^{\rm pa}\rightarrow\Lambda$ is determined by $\rho:\bar{\mathcal L}_\tau^{\rm pa}\rightarrow\Lambda$.

Here is a different way to define invariants $I_{\rm ss}(*,\tau)$ satisfying (85).

Definition 6.6. Let Assumption 3.5 hold, with $\mathbb{K}$ of characteristic zero. Then each $X\in\mathcal A$ may be written $X\cong X_1\oplus\cdots\oplus X_n$, for $X_1,\ldots,X_n$ *indecomposable* in $\mathcal A$, and unique up to order and isomorphism. Identifying X with $X_1\oplus\cdots\oplus X_n$, define an algebraic $\mathbb{K}$-subgroup T_X of $\operatorname{Aut}(X)$ by

$$T_X=\bigl\{\lambda_1\,\mathrm{id}_{X_1}+\lambda_2\,\mathrm{id}_{X_2}+\cdots+\lambda_n\,\mathrm{id}_{X_n}:\lambda_1,\ldots,\lambda_n\in\mathbb{K}^\times\bigr\}\cong(\mathbb{K}^\times)^n.$$

Then T_X is a maximal torus in $\operatorname{Aut}(X)$, so $\operatorname{Aut}(X)/T_X$ is a quasiprojective $\mathbb{K}$-variety, and its Euler characteristic $\chi(\operatorname{Aut}(X)/T_X)$ exists as in [27, §3.3].

Now $\operatorname{End}(X)$ is a finite-dimensional $\mathbb{K}$-algebra. Applying the *Wedderburn structure theorem* [3, §1.3] gives an algebra isomorphism $\operatorname{End}(X)/J(\operatorname{End}(X))\cong\bigoplus_{i=1}^r\operatorname{End}(\mathbb{K}^{n_i})$, where $J(\operatorname{End}(X))$ is the *Jacobson radical*, a nilpotent, two-sided ideal in $\operatorname{End}(X)$, and $n=n_1+\cdots+n_r$. This implies an isomorphism of varieties $\operatorname{Aut}(X)/T_X\cong\mathbb{K}^l\times\prod_{i=1}^n\mathrm{GL}(n_i,\mathbb{K})/(\mathbb{K}^\times)^{n_i}$, where $l=\dim J(\operatorname{End}(X))$ and $(\mathbb{K}^\times)^{n_i}\subseteq\mathrm{GL}(n_i,\mathbb{K})$ is the subgroup of diagonal matrices. Elementary calculation then shows that $\chi\bigl(\operatorname{Aut}(X)/T_X\bigr)=\prod_{i=1}^r n_i!$, which is *nonzero*.

Define $\Lambda=\mathbb{Q}[t]$, the $\mathbb{Q}$-algebra of rational polynomials in t. Define a *weight function* $w:\mathfrak{Obj}_{\mathcal A}(\mathbb{K})\rightarrow\Lambda$ by $w([X])=\chi\bigl(\operatorname{Aut}(X)/T_X\bigr)^{-1}t^n$, for n,T_X as above. This is well-defined as $\chi(\operatorname{Aut}(X)/T_X)\neq0$ from above, and is *locally constructible* on $\mathfrak{Obj}_{\mathcal A}$. One can prove that this weight function has the following multiplicative property. Let $\bullet$ be the partial order on $\{1,2\}$ with $i\bullet j$ only if $i=j$, and $P_{(\{1,2\},\bullet)}$ be the bilinear operation on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal A})$ studied in [30, §4.8]. Then for all $f,g\in\mathrm{CF}(\mathfrak{Obj}_{\mathcal A})$ we have

$$\chi^{\rm na}\bigl(\mathfrak{Obj}_{\mathcal A},w\cdot P_{(\{1,2\},\bullet)}(f,g)\bigr)=\chi^{\rm na}\bigl(\mathfrak{Obj}_{\mathcal A},w\cdot f\bigr)\cdot\chi^{\rm na}\bigl(\mathfrak{Obj}_{\mathcal A},w\cdot g\bigr)\quad\text{in }\Lambda,\tag{86}$$

using the *naïve weighted Euler characteristic* of [27, §4.1]. The proof is elementary: we calculate the multiples of $f([X])g([Y])$ contributing to each side at $[X\oplus Y]$ and show they are the same.

Now let $(\tau,T,\leqslant)$ be a permissible stability condition on $\mathcal A$ and define

$$\begin{aligned}I_{\rm ss}(I,\preceq,\kappa,\tau)&=\chi^{\rm na}\bigl(\mathfrak{Obj}_{\mathcal A},w\cdot\mathrm{CF}^{\rm stk}(\boldsymbol\sigma(I))\delta_{\rm ss}(I,\preceq,\kappa,\tau)\bigr)\\&=\chi^{\rm na}\bigl(\mathfrak{Obj}_{\mathcal A},w\cdot\pi^{\rm stk}_{\mathfrak{Obj}_{\mathcal A}}\circ\boldsymbol\sigma(I)_*\,\bar\delta_{\rm ss}(I,\preceq,\kappa,\tau)\bigr)\end{aligned}\tag{87}$$

for all $(I,\preceq,\kappa)$. Then using $\mathrm{CF}^{\rm stk}\bigl(\boldsymbol\sigma(I)\bigr)\delta_{\rm ss}(I,\preceq,\kappa,\tau)=P_{(I,\preceq)}\bigl(\delta_{\rm ss}^{\kappa(i)}(\tau):i\in I\bigr)$, (86) and [30, Th. 4.22] we find these invariants satisfy (85).

The $I_{\rm ss}(*,\tau)$ of (87) do not come from Definition 6.4. Remark 6.2(a) applies to them, so that invariants such as $I_{\rm st}^\alpha(\tau)$ in (83) have a clear interpretation. In fact, combining (14), (83) and (87) yields $I_{\rm st}^\alpha(\tau)=\chi^{\rm na}(\mathfrak{Obj}_{\mathcal A},w\cdot\delta_{\rm st}^\alpha(\tau))$. But any τ-stable X has $\mathrm{Aut}(X)\cong\mathbb{K}^\times$, so $w\cong t$ on $\mathrm{Obj}_{\rm st}^\alpha(\tau)$, giving $I_{\rm st}^\alpha(\tau)=t\,\chi^{\rm na}(\mathrm{Obj}_{\rm st}^\alpha(\tau))$, arguably the simplest, most obvious way to 'count' τ-stables.

6.2 When $\mathrm{Ext}^i(X,Y)=0$ for all $X,Y\in\mathcal A$ and $i>1$

Recall that a $\mathbb{K}$-linear abelian category $\mathcal A$ is called of *finite type* if $\mathrm{Ext}^i(X,Y)$ is a finite-dimensional $\mathbb{K}$-vector space for all $X,Y\in\mathcal A$ and $i\geqslant 0$, and $\mathrm{Ext}^i(X,Y)=0$ for $i\gg 0$. Then there is a unique biadditive map $\chi:K_0(\mathcal A)\times K_0(\mathcal A)\to\mathbb{Z}$ on the Grothendieck group $K_0(\mathcal A)$ known as the *Euler form*, satisfying

$$\chi\bigl([X],[Y]\bigr)=\textstyle\sum_{i\geqslant 0}(-1)^i\dim_{\mathbb K}\mathrm{Ext}^i(X,Y)\quad\text{for all }X,Y\in\mathcal A. \tag{88}$$

We shall suppose $K(\mathcal A)$ in Assumption 3.5 is chosen such that χ factors through the projection $K_0(\mathcal A)\to K(\mathcal A)$, and so descends to $\chi:K(\mathcal A)\times K(\mathcal A)\to\mathbb{Z}$. This holds for nearly all the examples of [29, §9–§10].

Now assume $\mathrm{Ext}^i(X,Y)=0$ for all $X,Y\in\mathcal A$ and $i>1$. Then (88) becomes

$$\dim_{\mathbb K}\mathrm{Hom}(X,Y)-\dim_{\mathbb K}\mathrm{Ext}^1(X,Y)=\chi\bigl([X],[Y]\bigr)\quad\text{for all }X,Y\in\mathcal A. \tag{89}$$

This happens for $\mathcal A=\mathrm{coh}(P)$ in [29, Ex. 9.1] with P a smooth projective curve, and for $\mathcal A=\text{mod-}\mathbb{K}Q$ in [29, Ex. 10.5].

Supposing (89) and Assumption 2.10, [30, §6.2] defined an *algebra morphism*

$$\Phi^\Lambda\circ\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal A}}:\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal A})\to A(\mathcal A,\Lambda,\chi), \tag{90}$$

where $A(\mathcal A,\Lambda,\chi)$ is an explicit algebra depending only on $K(\mathcal A),C(\mathcal A),\chi$ and Λ. In fact we defined Φ^Λ on the algebra $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal A},\Upsilon,\Lambda)$, but composing with the projection from $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal A})$ gives an algebra morphism from $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal A})$. Furthermore, we showed (90) intertwines the multilinear operations $P_{(I,\preceq)}$ on $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal A})$ of [30, §5.1] with operations $P_{(I,\preceq)}$ on $A(\mathcal A,\Lambda,\chi)$.

We shall define families of invariants $I_{\rm ss}^\alpha(\tau)^\Lambda$, $J^\alpha(\tau)^{\Lambda^\circ,\Omega}$ and $I_{\rm ss}(I,\preceq,\kappa,\tau)^\Lambda$, $J_{\rm si}^{\rm b}(I,\preceq,\kappa,\tau)^{\Lambda^\circ,\Omega}$ taking values in the algebras $\Lambda,\Lambda^\circ,\Omega$ of Assumption 2.10, which will satisfy multiplicative identities when (90) is an algebra morphism. For later use we define them without extra assumptions on $\mathrm{Ext}^i(X,Y)$.

Definition 6.7. Let Assumptions 2.10 and 3.5 hold and $(\tau,T,\leqslant)$ be a permissible weak stability condition. For all $\alpha\in C(\mathcal A)$ and $\mathcal A$-data $(I,\preceq,\kappa)$ define

$$I_{\rm ss}^\alpha(\tau)^\Lambda=\Upsilon'\circ\Pi_*\,\bar\delta_{\rm ss}^\alpha(\tau)\ \text{ and }\ I_{\rm ss}(I,\preceq,\kappa,\tau)^\Lambda=\Upsilon'\circ\Pi_*\circ\boldsymbol{\sigma}(I)_*\,\bar\delta_{\rm ss}(I,\preceq,\kappa,\tau) \tag{91}$$

in Λ, where $\Upsilon':\underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K})\to\Lambda$ is as in Theorem 2.13 and $\Pi:\mathfrak{Obj}_{\mathcal A}\to\mathrm{Spec}\,\mathbb{K}$ is the projection, so that Π_* maps $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal A})\to\underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K})$. Now suppose $(I,\preceq)$ is *connected*. Then [31, §8] defines stack functions $\bar\epsilon^\alpha(\tau),\bar\delta_{\rm si}^{\rm b}(I,\preceq,\kappa,\tau)$, which lie in $\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{Obj}_{\mathcal A})$ by [31, Th. 8.7]. Define

$$\begin{aligned}J^\alpha(\tau)^{\Lambda^\circ}&=(\ell-1)\Upsilon'\circ\Pi_*\,\bar\epsilon^\alpha(\tau)\qquad\text{and}\\ J_{\rm si}^{\rm b}(I,\preceq,\kappa,\tau)^{\Lambda^\circ}&=(\ell-1)\Upsilon'\circ\Pi_*\circ\boldsymbol{\sigma}(I)_*\,\bar\delta_{\rm si}^{\rm b}(I,\preceq,\kappa,\tau).\end{aligned} \tag{92}$$

Suppose $f \in \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, so that $\Pi_1^{\mathrm{vi}}(f) = f$. Then Π_1^{vi} is also the identity on the projection $\bar{\Pi}_{\mathrm{Spec}\,\mathbb{K}}^{\Upsilon,\Lambda^\circ} \circ \Pi_*(f)$ of f to $\underline{\mathrm{S}}\bar{\mathrm{F}}(\mathrm{Spec}\,\mathbb{K}, \Upsilon, \Lambda^\circ)$, since Π_1^{vi} commutes with $\bar{\Pi}_{\mathrm{Spec}\,\mathbb{K}}^{\Upsilon,\Lambda^\circ}$ and Π_*. Using the explicit description [28, Prop. 6.11] of $\underline{\mathrm{S}}\bar{\mathrm{F}}(\mathrm{Spec}\,\mathbb{K}, \Upsilon, \Lambda^\circ)$ we now see that $\bar{\Pi}_{\mathrm{Spec}\,\mathbb{K}}^{\Upsilon,\Lambda^\circ} \circ \Pi_*(f) = \beta[[\mathrm{Spec}\,\mathbb{K}/\mathbb{K}^\times]]$ for some $\beta \in \Lambda^\circ$. Now Υ' factors via $\bar{\Pi}_{\mathrm{Spec}\,\mathbb{K}}^{\Upsilon,\Lambda^\circ}$, and it easily follows that $(\ell-1)\Upsilon' \circ \Pi_*(f) = \beta$, so that $(\ell-1)\Upsilon' \circ \Pi_*$ maps $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}) \to \Lambda^\circ \subset \Lambda$. It follows that $J^\alpha(\tau)^{\Lambda^\circ}, J_{\mathrm{si}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)^{\Lambda^\circ}$ actually lie in Λ°. Thus we may define

$$J^\alpha(\tau)^\Omega = \pi\bigl(J^\alpha(\tau)^{\Lambda^\circ}\bigr) \quad \text{and} \quad J_{\mathrm{si}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)^\Omega = \pi\bigl(J_{\mathrm{si}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)^{\Lambda^\circ}\bigr), \tag{93}$$

where $\pi : \Lambda^\circ \to \Omega$ is as in Assumption 2.10.

Note that Remark 6.2(b) applies, so that

$$I_{\mathrm{ss}}^\alpha(\tau)^\Lambda = \Upsilon'\bigl(\bigl[\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)\bigr]\bigr) \text{ and } I_{\mathrm{ss}}(I,\preceq,\kappa,\tau)^\Lambda = \Upsilon'\bigl(\bigl[\mathcal{M}_{\mathrm{ss}}(I,\preceq,\kappa,\tau)_{\mathcal{A}}\bigr]\bigr).$$

When $(\tau, T, \leqslant)$ is a stability condition we can also define $J_{\mathrm{st}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)^{\Lambda^\circ,\Omega}$ in the same way, using $\bar{\delta}_{\mathrm{st}}^{\mathrm{b}}(I,\preceq,\kappa,\tau) \in \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ from [31, §8]. With the obvious notation we have $J_{\mathrm{si}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)^{\Lambda^\circ} = (\ell-1) I_{\mathrm{si}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)^\Lambda$.

Theorem 6.8. *Let Assumptions 2.10 and 3.5 hold and $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be biadditive and satisfy* (89). *Then for all $\alpha \in C(\mathcal{A})$ and $\mathcal{A}$-data $(I,\preceq,\kappa)$ the following hold, with only finitely many nonzero terms:*

$$J^\alpha(\tau)^{\Lambda^\circ} = \sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}} \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))} \frac{(-1)^{n-1}(\ell-1)}{n} \prod_{i=1}^n I_{\mathrm{ss}}^{\kappa(i)}(\tau)^\Lambda, \tag{94}$$

$$I_{\mathrm{ss}}^\alpha(\tau)^\Lambda = \sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}} \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))} \frac{(\ell-1)^{-n}}{n!} \prod_{i=1}^n J^{\kappa(i)}(\tau)^{\Lambda^\circ}, \tag{95}$$

$$\text{and} \qquad I_{\mathrm{ss}}(I,\preceq,\kappa,\tau)^\Lambda = \ell^{-\sum_{i\neq j\in I:\ i\preceq j}\chi(\kappa(j),\kappa(i))} \cdot \textstyle\prod_{i\in I} I_{\mathrm{ss}}^{\kappa(i)}(\tau)^\Lambda. \tag{96}$$

Suppose $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ are permissible weak stability conditions on $\mathcal{A}$, the change from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ is globally finite, and there exists a weak stability condition $(\hat\tau,\hat T,\leqslant)$ on $\mathcal{A}$ dominating $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ with the change from $(\hat\tau,\hat T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ locally finite. Then for all $\alpha \in C(\mathcal{A})$ the following hold, with only finitely many nonzero terms:

$$\begin{aligned} I_{\mathrm{ss}}^\alpha(\tilde\tau)^\Lambda = \sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} & S(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau) \cdot \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))} \cdot \\ & \textstyle\prod_{i=1}^n I_{\mathrm{ss}}^{\kappa(i)}(\tau)^\Lambda, \end{aligned} \tag{97}$$

$$\begin{aligned} J^\alpha(\tilde\tau)^{\Lambda^\circ} = \sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} & U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde\tau) \cdot \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))} \cdot \\ & (\ell-1)^{1-n} \textstyle\prod_{i=1}^n J^{\kappa(i)}(\tau)^{\Lambda^\circ}. \end{aligned} \tag{98}$$

Proof. The definition of Φ^Λ in [30, §6.2] gives $\Phi^\Lambda\circ\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\bar\delta^\alpha_{\rm ss}(\tau)\bigr)=I^\alpha_{\rm ss}(\tau)^\Lambda a^\alpha$, $\Phi^\Lambda\circ\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\boldsymbol\sigma(I)_*\,\bar\delta_{\rm ss}(I,\preceq,\kappa,\tau)\bigr)=I_{\rm ss}(I,\preceq,\kappa,\tau)^\Lambda a^{\kappa(I)}$ and $\Phi^\Lambda\circ\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\epsilon^\alpha(\tau)\bigr)=(\ell-1)^{-1}J^\alpha(\tau)^{\Lambda^\circ}a^\alpha$, where a^α for $\alpha\in\bar C(\mathcal{A})$ are a Λ-basis for $A(\mathcal{A},\Lambda,\chi)$. Equations (94)–(95) and (97)–(98) follow from (21), (23), (45) and (49) respectively, the definition [30, Def. 6.3] of multiplication in $A(\mathcal{A},\Lambda,\chi)$, and the fact [30, Th. 6.4] that (90) is an algebra morphism. The powers of $\ell-1$ compensate for the factor $\ell-1$ in (92). There are only finitely many nonzero terms in each equation as this holds for (21), (23) (45), and (49). Equation (96) follows from $\boldsymbol\sigma(I)_*\,\bar\delta_{\rm ss}(I,\preceq,\kappa,\tau)=P_{(I,\preceq)}\bigl(\bar\delta_{\rm ss}^{\kappa(i)}(\tau):i\in I\bigr)$, the definition [30, Def. 6.3] of $P_{(I,\preceq)}$ in $A(\mathcal{A},\Lambda,\chi)$, and the fact [30, Th. 6.4] that (90) intertwines the $P_{(I,\preceq)}$ on $\mathrm{SF}_{\rm al}(\mathfrak{Obj}_{\mathcal{A}})$ and $A(\mathcal{A},\Lambda,\chi)$. $\square$

Equations (94)–(95) show that when (89) holds the systems of invariants $I^\alpha_{\rm ss}(\tau)^\Lambda$ and $J^\alpha(\tau)^{\Lambda^\circ}$ for $\alpha\in C(\mathcal{A})$ are equivalent. But one can argue the $J^\alpha(\tau)^{\Lambda^\circ}$ are better, since they take their values in the smaller algebra Λ°, and so it takes less information to describe them. Equivalently, the $I^\alpha_{\rm ss}(\tau)^\Lambda$ satisfy natural identities, that the right hand side of (94) lies in Λ° for all $\alpha\in C(\mathcal{A})$.

Suppose now that χ is *symmetric*, that is, $\chi(\alpha,\beta)=\chi(\beta,\alpha)$ for all $\alpha,\beta\in K(\mathcal{A})$. Then $A(\mathcal{A},\Lambda,\chi)$ is a *commutative* algebra, as its multiplication relations are $a^\alpha\star a^\beta=\ell^{-\chi(\beta,\alpha)}a^{\alpha+\beta}$. Now Theorem 5.4 shows that (49) may be rewritten

$$\bar\epsilon^\alpha(\tilde\tau)=\bar\epsilon^\alpha(\tau)+\text{sum of multiple commutators of two or more }\bar\epsilon^{\kappa(i)}(\tau).$$

When we project this to the commutative algebra $A(\mathcal{A},\Lambda,\chi)$ under the algebra morphism (90) the multiple commutators go to zero, giving $\Phi^\Lambda\circ\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\bar\epsilon^\alpha(\tilde\tau)\bigr)=\Phi^\Lambda\circ\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\bar\epsilon^\alpha(\tau)\bigr)$. So as $\Phi^\Lambda\circ\Pi^{\Upsilon,\Lambda}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\epsilon^\alpha(\tau)\bigr)=(\ell-1)^{-1}J^\alpha(\tau)^{\Lambda^\circ}a^\alpha$ we deduce:

Corollary 6.9. *In Theorem 6.8, if χ is symmetric then $J^\alpha(\tilde\tau)^{\Lambda^\circ}=J^\alpha(\tau)^{\Lambda^\circ}$ and $J^\alpha(\tilde\tau)^\Omega=J^\alpha(\tau)^\Omega$ for all $\alpha\in C(\mathcal{A})$.*

We compute the invariants of Definition 6.7 explicitly when $\mathcal{A}=$mod-$\mathbb{K}Q$.

Example 6.10. Let $Q=(Q_0,Q_1,b,e)$ be a *quiver*. That is, Q_0 is a finite set of *vertices*, Q_1 is a finite set of *arrows*, and $b,e:Q_1\rightarrow Q_0$ are maps giving the *beginning* and *end* of each arrow. Fix an algebraically closed field $\mathbb{K}$, and take $\mathcal{A}$ to be the abelian category mod-$\mathbb{K}Q$ of *representations* of Q. Objects $\mathbf{X}=(X_v,\rho_a)\in$ mod-$\mathbb{K}Q$ comprise of finite-dimensional $\mathbb{K}$-vector spaces X_v for all $v\in Q_0$ and linear maps $\rho_a:X_{b(a)}\rightarrow X_{e(a)}$ for all $a\in Q_1$.

Define data $K(\mathcal{A}),\mathfrak{F}_{\mathcal{A}}$ satisfying Assumption 3.5 as in [29, Ex. 10.5]. Then $K(\mathcal{A})=\mathbb{Z}^{Q_0}$, with elements of $K(\mathcal{A})$ written as maps $Q_0\rightarrow\mathbb{Z}$, and $[(X_v,\rho_a)]=\alpha\in K(\mathcal{A})$ if $\dim_{\mathbb{K}}X_v=\alpha(v)$ for all $v\in Q_0$. It is well-known that $\mathrm{Ext}^m(X,Y)=0$ for all $X,Y\in\mathcal{A}$ and $m\geqslant 2$, and (89) holds with χ given by

$$\chi(\alpha,\beta)=\textstyle\sum_{v\in Q_0}\alpha(v)\beta(v)-\sum_{a\in Q_1}\alpha(b(a))\beta(e(a))\quad\text{for }\alpha,\beta\in\mathbb{Z}^{Q_0}.$$

As in the proof of [29, Th. 10.11], for $\alpha\in C(\mathcal{A})$ there is a 1-isomorphism

$$\mathfrak{Obj}^\alpha_{\mathcal{A}}\cong\bigl[\mathbb{K}^{\sum_{a\in Q_1}\alpha(b(a))\alpha(e(a))}/\textstyle\prod_{v\in Q_0}\mathrm{GL}(\alpha(v),\mathbb{K})\bigr].\tag{99}$$

Now suppose Assumption 2.10 holds with this $\mathbb{K}$. Then Theorem 2.13 defines an algebra morphism $\Upsilon' : \underline{\mathrm{SF}}(\mathrm{Spec}\,\mathbb{K}) \to \Lambda$. Noting that $\prod_{v\in Q_0}\mathrm{GL}(\alpha(v),\mathbb{K})$ is a *special* algebraic $\mathbb{K}$-group and using Theorem 2.13, $\Upsilon([\mathbb{K}^n]) = \ell^n$ and a formula for $\Upsilon([\mathrm{GL}(m,\mathbb{K})])$ in [28, Lem. 4.5], we deduce from (99) that

$$\Upsilon'\bigl([\mathfrak{Obj}_{\mathcal{A}}^{\alpha}]\bigr) = \frac{\ell^{\sum_{a\in Q_1}\alpha(b(a))\alpha(e(a)) - \sum_{v\in Q_0}\alpha(v)(\alpha(v)-1)/2}}{\prod_{v\in Q_0}\prod_{k=1}^{\alpha(v)}\bigl(\ell^k-1\bigr)}. \tag{100}$$

Let $(\tau,T,\leqslant)$ be any weak stability condition on $\mathcal{A}$, such as the slope stability conditions of [29, Ex. 4.14]. Then $(\tau,T,\leqslant)$ is permissible by [29, Cor. 4.13]. Define invariants $I_{\mathrm{ss}}^{\alpha}(\tau)^{\Lambda}, I_{\mathrm{ss}}(I,\preceq,\kappa,\tau)^{\Lambda}$ as in (91). One possibility for $(\tau,T,\leqslant)$ is the trivial stability condition $(0,\{0\},\leqslant)$. Since every object is 0-semistable we have $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(0) = \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$, and thus $I_{\mathrm{ss}}^{\alpha}(\tau)^{\Lambda} = \Upsilon'\bigl([\mathfrak{Obj}_{\mathcal{A}}^{\alpha}]\bigr)$, which is given by (100). Applying Theorem 6.8 with $(0,\{0\},\leqslant)$ in place of $(\tau,T,\leqslant),(\hat\tau,\hat T,\leqslant)$ and $(\tau,T,\leqslant)$ in place of $(\tilde\tau,\tilde T,\leqslant)$, and simplifying $S(\{1,\ldots,n\},\leqslant,\kappa,0,\tau)$ using (60), from (97) and (100) we deduce that for all $\alpha\in C(\mathcal{A})$ we have

$$\begin{aligned}\sum_{\substack{\mathcal{A}\text{-data}\\(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\\ \tau\circ\kappa(\{1,\ldots,i\})>\tau(\alpha),\\ 1\leqslant i<n}} (-1)^{n-1}\cdot &\Biggl[\prod_{1\leqslant i<j\leqslant n}\ell^{\sum_{a\in Q_1}\kappa(j)(b(a))\kappa(i)(e(a))-\sum_{v\in Q_0}\kappa(j)(v)\kappa(i)(v)}\Biggr]\cdot\\ &\Biggl[\prod_{i=1}^{n}\frac{\ell^{\sum_{a\in Q_1}\kappa(i)(b(a))\kappa(i)(e(a))-\sum_{v\in Q_0}\kappa(i)(v)(\kappa(i)(v)-1)/2}}{\prod_{v\in Q_0}\prod_{k=1}^{\kappa(i)(v)}\bigl(\ell^k-1\bigr)}\Biggr]\\ &= I_{\mathrm{ss}}^{\alpha}(\tau)^{\Lambda}.\end{aligned} \tag{101}$$

Combining this with (94) and (96) gives expressions for $J^{\alpha}(\tau)^{\Lambda^{\circ}}, I_{\mathrm{ss}}(I,\preceq,\kappa,\tau)^{\Lambda}$. It might be interesting rewrite this formula for $J^{\alpha}(\tau)^{\Lambda^{\circ}}$ so that every term lies in Λ°. Then projecting to Ω would give a formula for $J^{\alpha}(\tau)^{\Omega}$. This would, for instance, enable easier calculation of Euler characteristics of quiver moduli spaces in the case $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau) = \mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)$.

We can relate Example 6.10 to results of Reineke [38]. Take $\mathbb{K}=\mathbb{C}$, let $(\tau,T,\leqslant)$ be a slope stability condition on $\mathcal{A} = \text{mod-}\mathbb{C}Q$ as in [31, Ex. 4.14], and let $\alpha\in C(\mathcal{A})$ be 'coprime', which essentially means that $\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau) = \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$, that is, τ-stability and τ-semistability coincide in class α. Regarding $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ as an open $\mathbb{C}$-substack of $\mathfrak{Obj}_{\mathcal{A}}$ we have a 1-isomorphism $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)\cong\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)\times[\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^{\times}]$, where $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$ is a *nonsingular complex projective variety*. Here the factor $[\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^{\times}]$ arises as $\mathrm{Iso}_{\mathbb{C}}(X)\cong\mathbb{C}^{\times}$ for all $X\in\mathrm{Obj}_{\mathrm{st}}^{\alpha}(\tau)=\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$, and we can split the stabilizer groups off as a product with $[\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^{\times}]$. Since $\Upsilon([\mathbb{C}^{\times}]) = \ell-1$, we see that

$$\Upsilon\bigl(\bigl[\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)\bigr]\bigr) = (\ell-1)\Upsilon'\bigl(\bigl[\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)\times[\mathrm{Spec}\,\mathbb{C}/\mathbb{C}^{\times}]\bigr]\bigr) = (\ell-1)I_{\mathrm{ss}}^{\alpha}(\tau)^{\Lambda}. \tag{102}$$

Equations (101) and (102) thus give an explicit expression for $\Upsilon([\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)])$.

Now in [38, Cor. 6.8], Reineke gives a formula for the Poincaré polynomial $P\bigl(\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau);z\bigr)$, as in Example 2.11. Putting $\ell=z^2$ in (101) and (102), careful comparison shows that our formula agrees with Reineke's. We have reproved

Reineke's result by quite different methods, and also extended it: Reineke restricts to $\mathbb{K} = \mathbb{C}$, to slope functions, to Poincaré polynomials, and to coprime α, but Example 6.10 holds for algebraically closed $\mathbb{K}$, arbitrary weak stability conditions, any motivic invariant Υ satisfying Assumption 2.10 (such as virtual Hodge polynomials), and all $\alpha \in C(\mathcal{A})$. In particular, we can now interpret Reineke's formula for non-coprime α.

6.3 Counting vector bundles on smooth curves

In two seminal papers which have been elaborated on by many other authors since, Harder and Narasimhan [21] and Atiyah and Bott [1] found recursive formulae for the Poincaré polynomials of moduli spaces of semistable vector bundles with fixed rank and determinant over a smooth curve P of genus g. We will now recast this in our own framework, using the ideas of §2.4.

Let $\mathbb{K}$ be an algebraically closed field, P a smooth projective curve over $\mathbb{K}$ with genus g, and take $\mathcal{A} = \mathrm{coh}(P)$ with data $K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$ satisfying Assumption 3.5 as in [29, Ex. 9.1]. Identify $K(\mathcal{A}) = \mathbb{Z}^2$, such that if X is a vector bundle over P with rank n and degree d then $[X] = (n, d)$ in $K(\mathcal{A}) = \mathbb{Z}^2$, and then

$$C(\mathcal{A}) = \big\{(n, d) \in \mathbb{Z}^2 : n \geqslant 0, \text{ and } d > 0 \text{ if } n = 0\big\}. \tag{103}$$

Also $\mathrm{Ext}^i(X, Y) = 0$ for all $X, Y \in \mathcal{A}$ and $i > 1$, so (89) holds, and using the Riemann–Roch Theorem we find that $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}^2$ is given by

$$\chi\big((n_1, d_1), (n_2, d_2)\big) = n_1 d_2 - d_1 n_2 - (g - 1) n_1 n_2. \tag{104}$$

There is an ample line bundle E on P with $[E] = (1, 1)$ in $K(\mathcal{A})$, such that if $X \in \mathcal{A}$ with $[X] = (n, d)$ in $K(\mathcal{A})$ then the Hilbert polynomial of X w.r.t. E is $p_X(t) = nt + d + n(1 - g)$. Let $(\gamma, G_1, \leqslant)$ be the Gieseker stability condition on $\mathcal{A} = \mathrm{coh}(P)$ defined in [31, Ex. 4.16] using this E; since $\dim P = 1$ this coincides with μ-stability in [31, Ex. 4.17]. It is *permissible* by [31, Th. 4.20]. Let $(\delta, D_1, \leqslant)$ be the *purity* weak stability condition on $\mathcal{A}$ defined in [31, Ex. 4.18], so that $X \in \mathcal{A}$ is δ-semistable if and only if it is pure. It is *not* permissible.

As $(\delta, D_1, \leqslant)$ dominates $(\gamma, G_1, \leqslant)$ Theorem 5.11 applies with $(\gamma, G_1, \leqslant)$, $(\delta, D_1, \leqslant)$ in place of $(\tau, T, \leqslant), (\tilde\tau, \tilde T, \leqslant)$. We shall rewrite (57) with $\alpha = (n, d)$ for $n > 0$ and $d \in \mathbb{Z}$. For n, κ as in (57), replace n by k and write $\kappa(i) = (n_i, d_i)$ for $i = 1, \ldots, k$, and use (103). Since $n > 0$ we have $\delta(\alpha) = t$, and from [31, Ex. 4.18] we see that $\delta \circ \kappa(i)$ is t if $n_i > 0$ and 1 if $n_i = 0$, so the condition $\tilde\tau \circ \kappa \equiv \tilde\tau(\alpha)$ in (57) is equivalent to $n_i > 0$ for all i. Then $\gamma \circ \kappa(i) = t + d_i/n_i + 1 - g$, so $\tau \circ \kappa(1) > \cdots > \tau \circ \kappa(n)$ in (57) is equivalent to $d_1/n_1 > d_2/n_2 > \cdots > d_k/n_k$. Putting all this together, Theorem 5.11 shows that

$$\begin{aligned}\sum_{k=1}^{n} \sum_{\substack{n_1,\ldots,n_k>0:\\ n_1+\cdots+n_k=n}} \sum_{\substack{d_1,\ldots,d_k\in\mathbb{Z}:\ d_1+\cdots+d_k=d,\\ d_1/n_1>\cdots>d_k/n_k}} &\bar\delta_{\mathrm{ss}}^{(n_1,d_1)}(\gamma) * \bar\delta_{\mathrm{ss}}^{(n_2,d_2)}(\gamma) * \cdots * \bar\delta_{\mathrm{ss}}^{(n_k,d_k)}(\gamma)\\ &= \bar\delta_{\mathrm{ss}}^{(n,d)}(\delta),\end{aligned} \tag{105}$$

which holds as an infinite sum in $\mathrm{LSF}(\mathfrak{Obj}_{\mathcal{A}})$ converging as in Definition 2.16.

We shall show that (105) is *strongly convergent*. To do this we will need some dimension calculations. Suppose $\alpha \in C(\mathcal{A})$ and $[X] \in \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\gamma)$. Then the Zariski tangent space to $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ at $[X]$ is $\mathrm{Ext}^1(X,X)$, and $\mathrm{Iso}_{\mathbb{K}}(X) = \mathrm{Aut}(X)$ which is open in $\mathrm{Hom}(X,X)$. Since $\mathrm{Ext}^2(X,X) = 0$ and $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\gamma)$ is open in $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ we see by deformation theory (see for instance [23, §2.A, §4.5]) that the dimension of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\gamma)$ near X is $\dim \mathrm{Ext}^1(X,X) - \dim \mathrm{Hom}(X,X)$, which is $-\chi(\alpha,\alpha)$ by (89). As this is independent of $[X]$ we see that $\dim \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\gamma) = \dim \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\delta) = -\chi(\alpha,\alpha)$ provided $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\gamma), \mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\delta) \neq \emptyset$. Therefore

$$\bar{\delta}_{\mathrm{ss}}^{\alpha}(\gamma) \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})_{-\chi(\alpha,\alpha)} \quad \text{and} \quad \bar{\delta}_{\mathrm{ss}}^{\alpha}(\delta) \in \mathrm{LSF}(\mathfrak{Obj}_{\mathcal{A}})_{-\chi(\alpha,\alpha)}. \tag{106}$$

Lemma 6.11. *In the situation above, suppose $a,b \in \mathbb{Z}$, $\alpha,\beta \in C(\mathcal{A})$ and $f \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})_a$ and $g \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})_b$ are supported on $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K})$ and $\mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K})$ respectively. Then $f * g \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})_{a+b-\chi(\beta,\alpha)}$.*

Proof. Clearly $f \otimes g \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}})_{a+b}$. Now [30, Cor. 5.15] gives explicit expressions for $f \otimes g$ and $f * g$, involving vector spaces E_m^0, E_m^1 isomorphic to $\mathrm{Hom}(Y,X)$ and $\mathrm{Ext}^1(Y,X)$ for $([X],[Y])$ in the support of $f \otimes g$. As $([X],[Y]) \in \mathfrak{Obj}_{\mathcal{A}}^{\alpha}(\mathbb{K}) \times \mathfrak{Obj}_{\mathcal{A}}^{\beta}(\mathbb{K})$ we see from (89) that $\dim E_m^1 - \dim E_m^0 = -\chi(\beta,\alpha)$. The lemma then easily follows from [30, Cor. 5.15]. □

If $(n_i,d_i) \in C(\mathcal{A})$ for $i = 1,\ldots,k$ with $n = n_1 + \cdots + n_k$ and $d = d_1 + \cdots + d_k$ then using (104), (106), induction on k and biadditivity of χ we see that

$$\begin{aligned}
&\bar{\delta}_{\mathrm{ss}}^{(n_1,d_1)}(\gamma) * \bar{\delta}_{\mathrm{ss}}^{(n_2,d_2)}(\gamma) * \cdots * \bar{\delta}_{\mathrm{ss}}^{(n_k,d_k)}(\gamma) \in \Lambda_N, \quad \text{where}\\
N &= -\textstyle\sum_{i=1}^{k} \chi\big((n_i,d_i),(n_i,d_i)\big) - \sum_{1 \leqslant i<j \leqslant k} \chi\big((n_j,d_j),(n_i,d_i)\big)\\
&= -\chi\big((n,d),(n,d)\big) + \textstyle\sum_{1 \leqslant i<j \leqslant k} \chi\big((n_i,d_i),(n_j,d_j)\big)\\
&= (g-1)n^2 - (g-1) \textstyle\sum_{1 \leqslant i<j \leqslant k} n_i n_j + \sum_{1 \leqslant i<j \leqslant k} (n_i d_j - d_i n_j)\\
&\leqslant \max\big(0,(g-1)\big)n^2 + \textstyle\sum_{1 \leqslant i<j \leqslant k} (n_i d_j - d_i n_j).
\end{aligned} \tag{107}$$

For n_i, d_i as in (105) we have $d_1/n_1 > \cdots > d_k/n_k$, which implies that all terms $n_i d_j - d_i n_j$ in the bottom line of (107) are *negative*. Using this we prove:

Proposition 6.12. *Equation* (105) *is strongly convergent, in the sense of Definition 2.16. Hence $\bar{\delta}_{\mathrm{ss}}^{(n,d)}(\delta) \in \mathrm{ESF}(\mathfrak{Obj}_{\mathcal{A}})$ for all $n > 0$ and $d \in \mathbb{Z}$.*

Proof. We already know (105) is convergent. To prove it is strongly convergent, it is enough to show that for each $m \in \mathbb{Z}$ there are only finitely many choices of k and n_i, d_i in (105) with $N > m$ in (107). Let k, n_i, d_i be some such choice. As $n_i d_j - d_i n_j < 0$ for all $1 \leqslant i < j \leqslant k$ from above, we see from (107) that for all $1 \leqslant i < j \leqslant k$ we have

$$m - \max\big(0,(g-1)\big)n^2 < n_i d_j - d_i n_j < 0. \tag{108}$$

Clearly there are only finitely many choices for k and $n_1,\ldots,n_k$. For each such choice (108) shows there are only finitely many possibilities for the integers

$n_i d_j - d_i n_j$ for $1 \leqslant i < j \leqslant k$. But these and $d_1 + \cdots + d_k = d$ determine $d_1, \ldots, d_k$, so there are only finitely many choices for $d_1, \ldots, d_k$. This proves (105) is strongly convergent. The final part follows as $\mathrm{ESF}(\mathfrak{Obj}_{\mathcal{A}})$ is closed under strongly convergent limits, from Definition 2.16. $\square$

If $d > 0$, so that $(0, d) \in C(\mathcal{A})$, it is easy to show that every $[X] \in \mathfrak{Obj}_{\mathcal{A}}(\mathbb{K})$ is γ- and δ-semistable, so that $\mathrm{Obj}_{\mathrm{ss}}^{(0,d)}(\gamma) = \mathrm{Obj}_{\mathrm{ss}}^{(0,d)}(\delta) = \mathfrak{Obj}_{\mathcal{A}}^{(0,d)}(\mathbb{K})$. Therefore $\bar{\delta}_{\mathrm{ss}}^{(0,d)}(\delta) = \bar{\delta}_{\mathrm{ss}}^{(0,d)}(\gamma) \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, as $(\gamma, G_1, \leqslant)$ is permissible. Together with Proposition 6.12 this shows $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\delta) \in \mathrm{ESF}(\mathfrak{Obj}_{\mathcal{A}})$ for all $\alpha \in C(\mathcal{A})$.

Definition 6.13. Let Assumption 3.5 hold and $(\tau, T, \leqslant)$ be a weak stability condition on $\mathcal{A}$. Generalizing Definition 3.15, we call $(\tau, T, \leqslant)$ *essentially permissible* if (i) $\mathcal{A}$ is τ-artinian and (ii) $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau) \in \mathrm{ESF}(\mathfrak{Obj}_{\mathcal{A}})$ for all $\alpha \in C(\mathcal{A})$. Clearly, $(\tau, T, \leqslant)$ permissible implies $(\tau, T, \leqslant)$ essentially permissible.

When $\mathcal{A} = \mathrm{coh}(P)$ for P a smooth curve as above, part (i) holds for $(\delta, D_1, \leqslant)$ by [31, Lem. 4.19], and (ii) holds as above. Thus $(\delta, D_1, \leqslant)$ is essentially permissible, but not permissible.

Suppose Assumptions 2.17 and 3.5 hold and $(\tau, T, \leqslant)$ is an essentially permissible weak stability condition on $\mathcal{A}$. For $\alpha \in C(\mathcal{A})$, define *invariants* $I_{\mathrm{ss}}^{\alpha}(\tau)^{\Lambda}$ in Λ by $I_{\mathrm{ss}}^{\alpha}(\tau)^{\Lambda} = \Pi_{\Lambda}(\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau))$, where Π_{Λ} is as in Definition 2.22. If $(\tau, T, \leqslant)$ is permissible then $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\tau) \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, and so the $I_{\mathrm{ss}}^{\alpha}(\tau)^{\Lambda}$ agree with those defined in (91), as Π_{Λ} coincides with $\Upsilon' \circ \Pi_*$ on $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ by Definition 2.22.

Combining Propositions 2.23 and 6.12 shows that applying Π_{Λ} to (105) yields a convergent identity in Λ. Since (89) holds we are in the situation of §6.2, and as $\bar{\delta}_{\mathrm{ss}}^{(n_i,d_i)}(\gamma) \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ and Π_{Λ} coincides with $\Upsilon' \circ \Pi_*$ on $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, as in the proof of (97) we can rewrite $\Pi_{\Lambda}\bigl(\bar{\delta}_{\mathrm{ss}}^{(n_1,d_1)}(\gamma) * \cdots * \bar{\delta}_{\mathrm{ss}}^{(n_k,d_k)}(\gamma)\bigr)$ as a power of ℓ times the product of the $\Pi_{\Lambda}\bigl(\bar{\delta}_{\mathrm{ss}}^{(n_i,d_i)}(\gamma)\bigr)$. This proves:

Corollary 6.14. *Suppose Assumption 2.17 holds, $\mathcal{A} = \mathrm{coh}(P)$ for P a smooth projective curve over $\mathbb{K}$ of genus g, and $(\gamma, G_1, \leqslant), (\delta, D_1, \leqslant)$ are as above. Let $I_{\mathrm{ss}}^{\alpha}(\gamma)^{\Lambda}, I_{\mathrm{ss}}^{\alpha}(\delta)^{\Lambda}$ be as in Definition 6.13. Then for all $n > 0$ and $d \in \mathbb{Z}$ we have*

$$\sum_{k=1}^{n} \sum_{\substack{n_1,\ldots,n_k>0:\\ n_1+\cdots+n_k=n}} \sum_{\substack{d_1,\ldots,d_k\in\mathbb{Z}:\ d_1+\cdots+d_k=d,\\ d_1/n_1>\cdots>d_k/n_k}} \begin{aligned}[t] &\ell^{\sum_{1\leqslant i<j\leqslant n}(n_i d_j - d_i n_j + (g-1)n_i n_j)}\cdot \\ &\textstyle\prod_{i=1}^{k} I_{\mathrm{ss}}^{(n_i,d_i)}(\gamma)^{\Lambda} = I_{\mathrm{ss}}^{(n,d)}(\delta)^{\Lambda}, \end{aligned} \tag{109}$$

which holds as an infinite convergent sum in Λ, as in Definition 2.20.

In the same way, (62) implies the inverse identity to (105). Proposition 6.12 and (106) imply that $\bar{\delta}_{\mathrm{ss}}^{\alpha}(\delta) \in \mathrm{ESF}(\mathfrak{Obj}_{\mathcal{A}})_{-\chi(\alpha,\alpha)}$ for all $\alpha \in C(\mathcal{A})$, and using this and a similar argument to Proposition 6.12 we can show this inverse identity is strongly convergent. Applying Π_{Λ} as above thus shows:

Proposition 6.15. *In Corollary 6.14, for $n > 0$ and $d \in \mathbb{Z}$ we have*

$$\sum_{k=1}^{n} \sum_{\substack{n_1,\ldots,n_k>0:\\ n_1+\cdots+n_k=n}} \sum_{\substack{d_1,\ldots,d_k\in\mathbb{Z}:\ d_1+\cdots+d_k=d,\\ (d_1+\cdots+d_i)/(n_1+\cdots+n_i)>d/n,\ 1\leqslant i<k}} \begin{aligned}[t] &(-1)^{k-1}\ell^{\sum_{1\leqslant i<j\leqslant n}(n_i d_j - d_i n_j + (g-1)n_i n_j)}\cdot \\ &\textstyle\prod_{i=1}^{k} I_{\mathrm{ss}}^{(n_i,d_i)}(\delta)^{\Lambda} \\ &= I_{\mathrm{ss}}^{(n,d)}(\gamma)^{\Lambda}, \end{aligned} \tag{110}$$

which holds as an infinite convergent sum in Λ, *as in Definition 2.20.*

We shall give an explicit expression for $I_{\rm ss}^{(n,d)}(\delta)^\Lambda$, using the following notation. For C as above, write $\mathrm{Jac}(C)$ for the *Jacobian* of C, an abelian variety parametrizing degree 0 line bundles on C which is topologically a torus T^{2g} when $\mathbb{K}=\mathbb{C}$. For $m\geqslant 0$ write $C^{(m)}$ for the $m^{\rm th}$ symmetric power of C, which is a nonsingular projective $\mathbb{K}$-variety of dimension m. Then

Theorem 6.16. *In the situation above, for all* $n>0$ *and* $d\in\mathbb{Z}$ *we have*

$$I_{\rm ss}^{(n,d)}(\delta)^\Lambda=\frac{\Upsilon\bigl([\mathrm{Jac}(C)]\bigr)}{\ell-1}\sum_{m_2,\ldots,m_n\geqslant 0}\ell^{(n^2-1)(g-1)-\sum_{a=2}^n am_a}\prod_{a=2}^n\Upsilon\bigl([C^{(m_a)}]\bigr). \tag{111}$$

When Υ,Λ *are virtual Poincaré series as in Example 2.18 this simplifies to*

$$I_{\rm ss}^{(n,d)}(\delta)^\Lambda=\frac{(z+1)^{2g}(z^3+1)^{2g}\cdots(z^{2n-1}+1)^{2g}}{(z^2-1)^2(z^4-1)^2\cdots(z^{2n-2}-1)^2(z^{2n}-1)}\,. \tag{112}$$

Here (111) *converges in* Λ, *and the rational functions in* ℓ,z *in* (111)–(112) *may be interpreted as elements of* Λ *by writing them as power series in* ℓ^{-1},z^{-1}.

Equation (111) may be deduced from Behrend and Dhillon [2, §6], who using important results of Bifet, Ghione and Letizia [5] and Bialynicki-Birula [4] perform essentially the same calculation, except that they fix the determinants of their vector bundles; allowing determinants to vary gives our extra factor $\Upsilon\bigl([\mathrm{Jac}(C)]\bigr)$. Equation (112) comes from (111) using the following Poincaré polynomial formulae, the second due to MacDonald:

$$P\bigl(\mathrm{Jac}(C);z\bigr)=(1+z)^{2g}\quad\text{and}\quad\sum_{m=0}^\infty P\bigl(S^{(m)}C;z\bigr)t^m=\frac{(1+tz)^{2g}}{(1-t)(1-tz^2)}\,.$$

Substituting (111) or (112) into (110) then gives an explicit expression for $I_{\rm ss}^{(n,d)}(\gamma)^\Lambda$ which 'counts' semistable vector bundles in class (n,d).

We now explain how the above relates to work by other authors. Let $n\geqslant 1$ and $d\in\mathbb{Z}$ be coprime, and fix a line bundle L over P of degree d. Then there exists a moduli space $\mathcal{N}_{g,n,d}$ of semistable rank n vector bundles over P with determinant L, which is a smooth projective $\mathbb{K}$-variety, so its Poincaré polynomial $P(\mathcal{N}_{g,n,d};z)$ is well-defined. By counting semistable bundles over finite fields $\mathbb{F}_p$ and applying Deligne's solution of the Weil conjectures, Harder and Narasimhan [21] proved results on Betti numbers of $\mathcal{N}_{g,n,d}$, that were strengthened by Desale and Ramanan [15] to a recursive formula for $P(\mathcal{N}_{g,n,d};z)$, when $\mathbb{K}$ is an algebraic closure of $\mathbb{F}_p$.

These relate to our invariants for Υ,Λ as in Example 2.18 by

$$I_{\rm ss}^{(n,d)}(\gamma)^\Lambda=\frac{(1+z)^{2g}}{z^2-1}\,P(\mathcal{N}_{g,n,d};z),$$

where $(1+z)^{2g}$ compensates for us not fixing determinants, and $(z^2-1)^{-1}$ for the stabilizer groups $\mathbb{K}^\times$ of points in $\mathcal{N}_{g,n,d}$, which appear in our stack framework

but not in the moduli schemes context. Then Desale and Ramanan's recursive formula is equivalent to (109) above, with (112) substituted in for $I_{\mathrm{ss}}^{(n,d)}(\delta)^{\Lambda}$.

Atiyah and Bott [1] also derive recursive formulae for $P(\mathcal{N}_{g,n,d};z)$ when $\mathbb{K}=\mathbb{C}$, by completely different methods involving the topology of infinite-dimensional spaces. Their formula is equivalent to (109), but in a surprising way. As they explain on [1, p. 596], their formula is an infinite sum in *positive* powers of z. When n,d are coprime $\mathcal{N}_{g,n,d}$ is a nonsingular projective $\mathbb{K}$-variety of dimension $(g-1)(n^2-1)$, and so Poincaré duality implies that

$$P(\mathcal{N}_{g,n,d};z)=z^{2(g-1)(n^2-1)}P(\mathcal{N}_{g,n,d};z^{-1}). \tag{113}$$

If we replace z by z^{-1} throughout and use (113) to transform $P(\mathcal{N}_{g,n,d};z^{-1})$ into $P(\mathcal{N}_{g,n,d};z)$ then the Atiyah–Bott formulae again become equivalent to (109) above, with $\ell=z^2$ and (112) substituted in for $I_{\mathrm{ss}}^{(n,d)}(\delta)^{\Lambda}$.

This suggests an interesting explanation for why we are dealing with infinite series in negative powers of ℓ and z. If V is a smooth $\mathbb{C}$-variety of dimension d then we have (compactly-supported) cohomology groups $H^k(V,\mathbb{C})$ and $H^k_{\mathrm{cs}}(V,\mathbb{C})$ for $0\leqslant k\leqslant 2d$, with $H^k_{\mathrm{cs}}(V,\mathbb{C})\cong H^{2d-k}(V,\mathbb{C})^*$ by Poincaré duality. If $\mathfrak{F}$ is a smooth $\mathbb{C}$-stack of pure dimension d we also have cohomology groups $H^k(\mathfrak{F},\mathbb{C})$ for all $k\geqslant 0$, which can be nonzero for $k>2d$.

There may be a notion of *compactly-supported cohomology* $H^k_{\mathrm{cs}}(\mathfrak{F},\mathbb{C})$ *of stacks* with $H^k_{\mathrm{cs}}(\mathfrak{F},\mathbb{C})\cong H^{2d-k}(\mathfrak{F},\mathbb{C})^*$ for $\mathfrak{F}$ smooth of pure dimension d, which could be nonzero for integers $k\leqslant 2d$, and in particular for *negative* k. Compactly-supported cohomology is the right kind for our purposes, as it behaves in a motivic way and is used to define virtual Poincaré polynomials. So we can think of the negative powers of ℓ,z in our formulae as measuring some kind of compactly-supported cohomology of stacks which exists in negative dimensions.

6.4 Counting sheaves on surfaces P with K_P^{-1} nef

Let $\mathbb{K}$ be an algebraically closed field and P a smooth projective surface over $\mathbb{K}$. Take $\mathcal{A}=\mathrm{coh}(P)$ with data $K(\mathcal{A}),\mathfrak{F}_{\mathcal{A}}$ satisfying Assumption 3.5 as in [29, Ex. 9.1]. Then $\mathrm{Ext}^i(X,Y)=0$ for all $i>2$ and $X,Y\in\mathcal{A}$, so from (88) there is a biadditive $\chi:K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$ such that for all $X,Y\in\mathcal{A}$ we have

$$\dim_{\mathbb{K}}\mathrm{Hom}(X,Y)-\dim_{\mathbb{K}}\mathrm{Ext}^1(X,Y)+\dim_{\mathbb{K}}\mathrm{Ext}^2(X,Y)=\chi\bigl([X],[Y]\bigr). \tag{114}$$

Also, by Serre duality, writing K_P for the canonical bundle of P we have

$$\mathrm{Ext}^2(X,Y)\cong\mathrm{Hom}(Y,X\otimes K_P)^*\quad\text{for all }X,Y\in\mathcal{A}. \tag{115}$$

Let $(\gamma,G_2,\leqslant)$ be the Gieseker stability condition on $\mathcal{A}$ defined in [31, Ex. 4.16] using an ample line bundle E on P. It is permissible by [31, Th. 4.20]. Let $(\delta,D_2,\leqslant)$ be the purity weak stability condition on $\mathcal{A}$ defined in [31, Ex. 4.18].

First we consider how much of §6.3 can be generalized to the surface case. Unfortunately the answer is very little. The next example shows Proposition 6.12 does not generalize to $\mathbb{KP}^2$; a similar argument works for all surfaces P.

Example 6.17. Let $P = \mathbb{KP}^2$. Computations with Chern classes show we may identify $K(\mathcal{A})$ with $\{(a,b,c) \in \mathbb{Q}^3 : a, b, c - \frac{1}{2}b \in \mathbb{Z}\}$ so that $[E] = \big(\mathrm{rk}(E), c_1(E), c_2(E) - \frac{1}{2}c_1(E)^2)\big)$ for $E \in \mathcal{A}$. Then χ is given explicitly by

$$\chi\big((a_1,b_1,c_1),(a_2,b_2,c_2)\big) = a_1a_2 + \tfrac{3}{2}(a_1b_2 - b_1a_2) - b_1b_2 + a_1c_2 + c_1a_2. \quad (116)$$

Suppose $V_n \in \mathcal{A}$ is a rank 2 vector bundle with $[V_n] = (2,0,0)$ in $C(\mathcal{A})$, such that $V_n \otimes \mathcal{O}(-n)$ has a section s vanishing transversely at finitely many points for some $n \geqslant 1$. Calculation with Chern classes show there must be n^2 such points $x_1, \ldots, x_{n^2}$, and as $[\mathcal{O}(n)] = (1, n, \frac{1}{2}n^2)$ in $C(\mathcal{A})$ there is a pure sheaf X_n with $[X_n] = (1, -n, -\frac{1}{2}n^2)$ in $C(\mathcal{A})$ fitting into short exact sequences

$$0 \to \mathcal{O}(n) \xrightarrow{s} V_n \to X_n \to 0 \quad \text{and} \quad 0 \to X_n \to \mathcal{O}(-n) \to \textstyle\bigoplus_{i=1}^{n^2} \mathcal{O}_{x_i}.$$

Here $\mathcal{O}(n) \subset V_n$ is the γ Harder–Narasimhan filtration of V_n, so V_n is γ-unstable. Clearly X_n is determined up to isomorphism by $x_1, \ldots, x_{n^2}$. We shall describe the family of such vector bundles V_n.

We have $\mathrm{Hom}(X_n, \mathcal{O}(n)) \cong \mathrm{Hom}(\mathcal{O}(-n), \mathcal{O}(n)) \cong H^0(\mathcal{O}(2n)) \cong \mathbb{K}^{2n^2+3n+1}$, and $\mathrm{Ext}^2(X_n, \mathcal{O}(n))^* \cong \mathrm{Hom}(\mathcal{O}(n), X_n \otimes \mathcal{O}(-3)) \subseteq \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(-n-3)) \cong H^0(\mathcal{O}(-2n-3)) = 0$ by (115), so $\mathrm{Ext}^2(X_n, \mathcal{O}(n)) = 0$. As $\chi([X_n], [\mathcal{O}(n)]) = n^2 + 3n + 1$ by (116) we see that $\mathrm{Ext}^1(X_n, \mathcal{O}(n)) \cong \mathbb{K}^{n^2}$. One can show $\mathrm{Ext}^1(X_n, \mathcal{O}(n))$ is the direct sum of a copy of $\mathbb{K}$ located at each x_i, and the extension of $\mathcal{O}(n)$ by X_n corresponding to an element of $\mathrm{Ext}^1(X_n, \mathcal{O}(n))$ is a vector bundle V_n if and only if the components in each of these n^2 copies of $\mathbb{K}$ are nonzero. We also find that $\mathrm{Aut}(V_n) \cong \mathbb{K}^\times \ltimes \mathrm{Hom}(X_n, \mathcal{O}(n))$, so that $\dim \mathrm{Aut}(V_n) = 2n^2 + 3n + 2$, and the family of all such vector bundles V_n has dimension $3n^2 - 1$, that is, $2n^2$ parameters for the choice of $x_1, \ldots, x_{n^2}$ in $\mathbb{KP}^2$, plus $n^2 - 1$ for the extensions $P(\mathrm{Ext}^1(X_n, \mathcal{O}(n)))$.

Thus the family of all such points $[V_n]$ is a $\mathbb{K}$-substack of $\mathrm{Obj}_{\mathrm{ss}}^{(2,0,0)}(\delta) \subset \mathfrak{Obj}_{\mathcal{A}}^{(2,0,0)}$, the pure sheaves in class $(2,0,0)$, with dimension $n^2 - 3n - 3$, that is, the naïve dimension $3n^2 - 1$ of the family minus the dimension $2n^2 + 3n + 2$ of the stabilizer groups. Since $n^2 - 3n - 3 \to \infty$ as $n \to \infty$ we have $\dim \mathrm{Obj}_{\mathrm{ss}}^{(2,0,0)}(\delta) = \infty$. This implies that $\bar{\delta}_{\mathrm{ss}}^{(2,0,0)}(\delta) \notin \mathrm{ESF}(\mathfrak{Obj}_{\mathcal{A}})$, so $(\delta, D_2, \leqslant)$ is *not* essentially permissible in the sense of Definition 6.13.

Because of this, for surfaces the invariants $I_{\mathrm{ss}}^\alpha(\delta)^\Lambda$ counting pure sheaves on P are undefined, and the analogues of (109)–(112) do not make sense. The example also shows that it will not help to work with vector bundles rather than pure sheaves, as the $\mathbb{K}$-substack of $\mathfrak{Obj}_{\mathcal{A}}$ of vector bundles in class $(2,0,0)$ on $\mathbb{KP}^2$ has dimension ∞ and is not essentially permissible either. So we cannot hope to define invariants counting all vector bundles in class α on a surface P in any meaningful sense.

However, note that if P is a *ruled surface* with ruling $\pi : P \to C$, Yoshioka [45] computes the Betti numbers of moduli spaces of stable rank 2 coherent sheaves on P in a similar way to Harder and Narasimhan for curves. His method involves counting over finite fields the number of sheaves in class α on P whose

restriction to generic fibres of π is semistable. By analogy with §6.3, the author expects that semistability on generic fibres of π comes from an essentially permissible stability condition $(\zeta, Z, \leqslant)$ on $\mathrm{coh}(P)$, and one could hope to prove an analogue of Theorem 6.16 evaluating the invariants $I_{\mathrm{ss}}^{\alpha}(\zeta)^{\Lambda}$.

In the rest of the section we will show that if the anticanonical bundle K_P^{-1} of P is *numerically effective* (*nef*) then the invariants $I_{\mathrm{ss}}^{\alpha}(\gamma)^{\Lambda}$ of (91) transform according to (97) under change of Gieseker stability condition, even though (90) is *not* an algebra morphism. The basic idea is that for K_P^{-1} nef we use (115) to force $\mathrm{Ext}^2(X,Y)=0$ for X,Y satisfying some conditions, so that (114) reduces to (89) for these X,Y.

The classification of algebraic surfaces, described for instance in Iskovskikh and Shafarevich [24], determines the possible surfaces P with K_P^{-1} nef very explicitly. As K_P^{-1} is nef either (a) K_P^n admits no sections for all $n>0$, or (b) K_P^n is trivial for some $n>0$. In case (a) P has *Kodaira dimension* $-\infty$, and so is either rational or ruled. If also K_P^{-1} is *ample* P is called a *Del Pezzo surface* (or a *Fano* 2*-fold*), and is $\mathbb{KP}^2$, or $\mathbb{KP}^1\times\mathbb{KP}^1$, or the blow-up of $\mathbb{KP}^2$ in d points, $1\leqslant d\leqslant 9$. In case (b), P has Kodaira dimension 0, and is a $K3$ surface, an Enriques surface, an abelian surface, or a bielliptic surface.

The next three results show how we will use K_P^{-1} nef.

Lemma 6.18. *In the situation above, with K_P^{-1} nef, for all $0\not\cong W\in\mathrm{coh}(P)$ we have $\gamma([W])\leqslant\gamma([W\otimes K_P^{-1}])$ in G_2.*

Proof. Using Chern classes and the Riemann–Roch Theorem as in §5.6 we compute the Hilbert polynomials of W and $W\otimes K_P^{-1}$ with respect to E as

$$\begin{aligned} p_W(t) &= \bigl(\tfrac{1}{2}\,\mathrm{rk}(W)c_1(E)^2\bigr)t^2+\bigl(c_1(W)c_1(E)+\tfrac{1}{2}\,\mathrm{rk}(W)c_1(E)(c_1(K_P)-c_1(E))\bigr)t\\ &\quad+\bigl(\tfrac{1}{12}\,\mathrm{rk}(W)(c_1(K_P)^2-2c_2(TP))+\tfrac{1}{2}c_1(W)c_1(TP)+\tfrac{1}{2}c_1(W)^2-c_2(W)\bigr),\\ p_{W\otimes K_P^{-1}}(t) &= p_W(t)-\bigl(\mathrm{rk}(W)c_1(E)c_1(K_P)\bigr)t-\bigl(c_1(W)c_1(K_P)-\tfrac{1}{2}\,\mathrm{rk}(W)c_1(K_P)^2\bigr). \end{aligned}$$

As E is ample and K_P^{-1} nef we can show that $c_1(E)c_1(K_P)\leqslant 0$ with equality if and only if $c_1(K_P)=0$. Also $\mathrm{rk}(W)\geqslant 0$, and if $\mathrm{rk}(W)=0$ then $c_1(W)c_1(K_P)\leqslant 0$ as K_P^{-1} is nef. Using this we can show that $p_{W\otimes K_P^{-1}}(t)-p_W(t)$ has smaller degree than $p_W(t)$, and has positive leading coefficient if it is nonzero. Since $\gamma([W]),\gamma([W\otimes K_P^{-1}])$ are these Hilbert polynomials divided by their leading coefficients, it follows that $\gamma([W])\leqslant\gamma([W\otimes K_P^{-1}])$ in the total order '$\leqslant$' on G_2 defined in [31, §4.4]. □

Proposition 6.19. *In the situation above, with K_P^{-1} nef, suppose $X,Y\in\mathrm{coh}(P)$ are γ-semistable with $\gamma([X])<\gamma([Y])$. Then $\mathrm{Ext}^2(X,Y)=0$.*

Proof. By (115) it is enough to show that $\mathrm{Hom}(Y,X\otimes K_P)=0$. Suppose for a contradiction that $\phi:Y\to X\otimes K_P$ is a nonzero morphism in $\mathcal{A}$, and let $W\in\mathcal{A}$ be the image of ϕ. Then $W\not\cong 0$ is a quotient object of Y, so $\gamma([Y])\leqslant\gamma([W])$ as Y is γ-semistable. Also W is a subobject of $X\otimes K_P$, so $W\otimes K_P^{-1}$ is a subobject of X, giving $\gamma([W\otimes K_P^{-1}])\leqslant\gamma([X])$ as X is γ-semistable. Putting

this together with Lemma 6.18 gives $\gamma([Y]) \leqslant \gamma([W]) \leqslant \gamma([W \otimes K_P^{-1}]) \leqslant \gamma([X])$, which contradicts $\gamma([X]) < \gamma([Y])$. $\square$

Proposition 6.20. *In the situation above with K_P^{-1} nef, let Assumption 2.10 hold, and define $I_{\rm ss}^\alpha(\gamma)^\Lambda \in \Lambda$ for $\alpha \in C(\mathcal{A})$ by $I_{\rm ss}^\alpha(\gamma)^\Lambda = \Upsilon' \circ \Pi_* \,\bar\delta_{\rm ss}^\alpha(\gamma)$, as in (91). Suppose $(\{1,\ldots,n\},\leqslant,\kappa)$ is $\mathcal{A}$-data with $\gamma\circ\kappa(1) > \cdots > \gamma\circ\kappa(n)$. Then*

$$\Upsilon' \circ \Pi_*\bigl(\bar\delta_{\rm ss}^{\kappa(1)}(\gamma) * \cdots * \bar\delta_{\rm ss}^{\kappa(n)}(\gamma)\bigr) = \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))} \textstyle\prod_{i=1}^n I_{\rm ss}^{\kappa(i)}(\tau)^\Lambda. \quad (117)$$

Proof. Let $[Y_i] \in \mathrm{Obj}_{\rm ss}^{\kappa(i)}(\gamma)$ for $i = 1,\ldots,n$. Then Proposition 6.19 and $\gamma\circ\kappa(1) > \cdots > \gamma\circ\kappa(n)$ give $\mathrm{Ext}^2(Y_j, Y_i) = 0$ for $1 \leqslant i < j \leqslant n$. Let $[X_i]$ lie in the support of $\bar\delta_{\rm ss}^{\kappa(1)}(\gamma) * \cdots * \bar\delta_{\rm ss}^{\kappa(i)}(\gamma)$. Using exact sequences and induction on i we can show that $\mathrm{Ext}^2(Y_j, X_i) = 0$ whenever $1 \leqslant i < j \leqslant n$. For example, when $i = 2$ there is an exact sequence $0 \to Y_1 \to X_2 \to Y_2 \to 0$ for $[Y_1],[Y_2]$ in the supports of $\bar\delta_{\rm ss}^{\kappa(1)}(\gamma), \bar\delta_{\rm ss}^{\kappa(2)}(\gamma)$. This induces an exact sequence $\cdots \to \mathrm{Ext}^2(Y_j, Y_1) \to \mathrm{Ext}^2(Y_j, X_2) \to \mathrm{Ext}^2(Y_j, Y_2) \to 0$. As $\mathrm{Ext}^2(Y_j, Y_1) = \mathrm{Ext}^2(Y_j, Y_2) = 0$ from above this gives $\mathrm{Ext}^2(Y_j, X_2) = 0$. More generally $\mathrm{Ext}^2(Y_j, X_i)$ is built from terms $\mathrm{Ext}^2(Y_j, Y_a) = 0$ for $a = 1,\ldots,i$, and so is zero.

Apply [30, Cor. 5.15] with $f = \bar\delta_{\rm ss}^{\kappa(1)}(\gamma) * \cdots * \bar\delta_{\rm ss}^{\kappa(j-1)}(\gamma)$ and $g = \bar\delta_{\rm ss}^{\kappa(j)}(\gamma)$. This gives expressions for $f \otimes g$ and $f * g$, involving vector spaces E_m^0, E_m^1 isomorphic to $\mathrm{Hom}(Y_j, X_{j-1}), \mathrm{Ext}^1(Y_j, X_{j-1})$ for $([X_{j-1}],[Y_j])$ in the support of $f \otimes g$. As $\mathrm{Ext}^2(Y_j, X_{j-1}) = 0$ and $[X_{j-1}] = \kappa(\{1,\ldots,j-1\})$, $[Y_j] = \kappa(j)$ in $C(\mathcal{A})$ for such X, Y we see from (114) that $\dim E_m^1 - \dim E_m^0 = -\sum_{i=1}^{j-1}\chi(\kappa(j),\kappa(i))$. Hence

$$\begin{aligned}&\Upsilon' \circ \Pi_*\bigl(\bar\delta_{\rm ss}^{\kappa(1)}(\gamma) * \cdots * \bar\delta_{\rm ss}^{\kappa(j)}(\gamma)\bigr) = \\ &\ell^{-\sum_{i=1}^{j-1}\chi(\kappa(j),\kappa(i))}\Upsilon' \circ \Pi_*\bigl(\bar\delta_{\rm ss}^{\kappa(1)}(\gamma) * \cdots * \bar\delta_{\rm ss}^{\kappa(j-1)}(\gamma)\bigr)\Upsilon' \circ \Pi_*\bigl(\bar\delta_{\rm ss}^{\kappa(j)}(\gamma)\bigr),\end{aligned}$$

by the proof of [30, Th. 6.1]. Equation (117) follows by induction on j. $\square$

We can now prove our main result on invariants counting sheaves on surfaces. First we give a sketch of why it is true, which may be more helpful than the actual proof. In the situation of the theorem, suppose $(\tilde\gamma, G_2, \leqslant)$ dominates $(\gamma, G_2, \leqslant)$. Then Theorem 5.11 applies, and (57) writes $\bar\delta_{\rm ss}^\alpha(\tilde\gamma)$ as a sum of $\bar\delta_{\rm ss}^{\kappa(1)}(\gamma) * \cdots * \bar\delta_{\rm ss}^{\kappa(n)}(\gamma)$ with $\gamma\circ\kappa(1) > \cdots > \gamma\circ\kappa(n)$. So applying $\Upsilon' \circ \Pi_*$ and using Proposition 6.20 writes $I_{\rm ss}^\alpha(\tilde\gamma)^\Lambda$ as a sum of $\ell^{\cdots}\prod_i I_{\rm ss}^{\kappa(i)}(\gamma)^\Lambda$, a special case of (118) and an analogue of (109).

We then invert these identities in Λ to write the $I_{\rm ss}^\alpha(\gamma)^\Lambda$ in terms of the $I_{\rm ss}^\beta(\tilde\gamma)^\Lambda$. Exchanging $\gamma, \tilde\gamma$ proves another special case of (118), when $(\gamma, G_2, \leqslant)$ dominates $(\tilde\gamma, G_2, \leqslant)$, an analogue of (110). It is important that this inversion is done in Λ rather than trying to apply $\Upsilon' \circ \Pi_*$ to (62), as in (62) we do not have $\gamma\circ\kappa(1) > \cdots > \gamma\circ\kappa(n)$ and so cannot use Proposition 6.20.

For the general case, suppose we could find a permissible weak stability condition $(\hat\gamma, G_2, \leqslant)$ dominating both $(\gamma, G_2, \leqslant)$ and $(\tilde\gamma, G_2, \leqslant)$. Then the two

special cases above allow us to write $I_{\rm ss}^\alpha(\tilde\gamma)^\Lambda$ in terms of the $I_{\rm ss}^\beta(\hat\gamma)^\Lambda$ and $I_{\rm ss}^\beta(\hat\gamma)^\Lambda$ in terms of the $I_{\rm ss}^{\kappa(i)}(\gamma)^\Lambda$, and substituting one in the other yields (118).

The problem in proving Theorem 6.21 is that we have no suitable $(\hat\gamma,G_2,\leqslant)$. The purity weak stability condition $(\delta,D_2,\leqslant)$ will not do as it is not permissible, nor even essentially permissible, so the $I_{\rm ss}^\beta(\delta)^\Lambda$ are undefined. So instead we introduce a 1-parameter family of stability conditions $(\gamma_s,G_2,\leqslant)$ for $s\in[0,1]$ with $\gamma_0=\gamma$ and $\gamma_1=\tilde\gamma$, in a similar way to the proof of Theorem 5.16. There are finitely many 'walls' in $[0,1]$ where the expression for $\bar\delta_{\rm ss}^\alpha(\gamma_s)$ in terms of the $\bar\delta_{\rm ss}^\beta(\gamma)$ changes. Jumping onto a wall s from a nearby point s' is like transforming to a dominant weak stability condition, so we can use the ideas above.

Theorem 6.21. *Suppose $\mathbb K$ is an algebraically closed field and P a smooth projective surface over $\mathbb K$ with K_P^{-1} numerically effective. Take $\mathcal A=\mathrm{coh}(P)$ with data $K(\mathcal A),\mathfrak F_{\mathcal A}$ satisfying Assumption 3.5 as in* [29, Ex. 9.1], *with biadditive $\chi: K(\mathcal A)\times K(\mathcal A)\rightarrow\mathbb Z$ satisfying* (114) *for all $X,Y\in\mathcal A$. Let $(\gamma,G_2,\leqslant),(\tilde\gamma,G_2,\leqslant)$ be Gieseker stability conditions on $\mathcal A$ defined using ample line bundles $E,\tilde E$ on P as in* [31, Ex. 4.16]. *Suppose Assumption 2.10 holds, and define invariants $I_{\rm ss}^\alpha(\gamma)^\Lambda,I_{\rm ss}^\alpha(\tilde\gamma)^\Lambda$ for $\alpha\in C(\mathcal A)$ as in* (91). *Then for all $\alpha\in C(\mathcal A)$ the following holds, with only finitely many nonzero terms:*

$$I_{\rm ss}^\alpha(\tilde\gamma)^\Lambda=\sum_{\substack{\mathcal A\text{-}data\ (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &S(\{1,\ldots,n\},\leqslant,\kappa,\gamma,\tilde\gamma)\cdot\ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))}\cdot\\ &\qquad\qquad\qquad\textstyle\prod_{i=1}^n I_{\rm ss}^{\kappa(i)}(\gamma)^\Lambda.\end{aligned}\tag{118}$$

Proof. For $X\in\mathrm{coh}(P)$ define the *joint Hilbert polynomial* q_X w.r.t. $E,\tilde E$ by

$$q_X(k,l)=\textstyle\sum_{i=0}^{\dim P}(-1)^i\dim_{\mathbb K}H^i\bigl(P,X\otimes E^k\otimes\tilde E^l\bigr)\quad\text{for }k,l\in\mathbb Z.$$

As for conventional Hilbert polynomials, one can show that

$$q_X(k,l)=\textstyle\sum_{a,b=0}^{\dim P}c_{a,b}k^al^b/a!b!\quad\text{for }c_{a,b}\in\mathbb Z,$$

where $c_{a,b}=0$ if $a+b>\dim X$ and $c_{a,b}>0$ if $a+b=\dim X$. Also q_X is additive in $[X]\in K(\mathcal A)$, so there is a unique group homomorphism $\Pi_{E,\tilde E}: K(\mathcal A)\rightarrow\mathbb Q[t,u]$ with $\Pi_{E,\tilde E}:[X]\mapsto q_X(t,u)$ for all $X\in\mathrm{coh}(P)$.

Define $\gamma_s:C(\mathcal A)\rightarrow G_2$ by $\gamma_s(\alpha)=L_{\alpha,s}^{-1}\Pi_{E,\tilde E}\bigl((1-s)t,st\bigr)$ for each $s\in[0,1]$, where $L_{\alpha,s}$ is the leading coefficient of $\Pi_{E,\tilde E}\bigl((1-s)t,st\bigr)$, which is a polynomial in t with positive leading coefficient. Then $\gamma_0=\gamma$ and $\gamma_1=\tilde\gamma$, and the proof in [31, §4.4] shows that $(\gamma_s,G_2,\leqslant)$ is a stability condition on $\mathcal A$. As $q_X(t,0),q_X(0,t)$ are the Hilbert polynomials of X w.r.t. $E,\tilde E$ respectively, we see that $\gamma_0=\gamma$ and $\gamma_1=\tilde\gamma$, so the γ_s for $s\in[0,1]$ interpolate between γ and $\tilde\gamma$. When $s=p/(p+q)$ for integers $p,q\geqslant 0$ with $p+q>0$ it is equivalent (after rescaling t) to Gieseker stability with respect to the ample line bundle $E^q\otimes\tilde E^p$. Thus $(\gamma_s,G_2,\leqslant)$ is permissible for all $s\in[0,1]\cap\mathbb Q$ by [31, Th. 4.20].

Each $\alpha\in C(\mathcal A)$ has a Hilbert polynomial $p_\alpha(t)=a_dt^d/d!+\cdots+a_0$ w.r.t. E, with $a_i\in\mathbb Z$ and $a_d>0$, where $d=\dim\alpha$. We shall prove (118) by induction on a_d. Here is our inductive hypothesis, for $r\geqslant 1$:

($*_r$) Suppose that for all $\alpha \in C(\mathcal{A})$ whose Hilbert polynomial $p_\alpha(t) = a_d t^d/d! + \cdots + a_0$ w.r.t. E has $0 < a_d \leqslant r$, equation (118) holds with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde\gamma$, with only finitely many nonzero terms, for all $s, s' \in [0,1] \cap \mathbb{Q}$.

From Corollary 5.8 and Theorems 5.2 and 5.9 we see that (45) holds in $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ with $\gamma_s, \gamma_{s'}$ for $s, s' \in [0,1] \cap \mathbb{Q}$ in place of $\tau, \tilde\tau$ with finitely many nonzero terms. This implies there are only finitely many nonzero terms in (118) with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde\gamma$, proving part of ($*_r$).

Let α, d, a_d, s, s' be as in ($*_r$), and n, κ as in (45) or (118) with $S(\{1,\ldots,n\}, \leqslant, \kappa, \gamma_s, \gamma_{s'}) \neq 0$. Then the Hilbert polynomials $p_{\kappa(i)}(t)$ are also of the form $a_d^i t^d/d! + \cdots + a_0^i$ for $a_d^i \geqslant 1$, and sum to $p_\alpha(t)$. Thus $\sum_{i=1}^n a_d^i = a_d \leqslant r$, which forces $n \leqslant r$. In particular, when $r = 1$ the only nonzero terms in (45) and (118) are $n = 1$ and $\kappa(1) = \alpha$. Thus (45) reduces to $\bar\delta_{\mathrm{ss}}^\alpha(\gamma_{s'}) = \bar\delta_{\mathrm{ss}}^\alpha(\gamma_s)$, so $I_{\mathrm{ss}}^\alpha(\gamma_{s'})^\Lambda = I_{\mathrm{ss}}^\alpha(\gamma_s)^\Lambda$, which is (118). This proves ($*_1$), giving the first step.

Suppose by induction that ($*_r$) holds for some $r \geqslant 1$, and let $\alpha \in C(\mathcal{A})$ with $p_\alpha(t) = a_d t^d/d! + \cdots + a_0$ and $a_d = r+1$. Using the methods of §5.6 we can show that there are only *finitely many* sets of $\mathcal{A}$-data $(\{1,\ldots,n\}, \leqslant, \kappa)$ with $\kappa(\{1,\ldots,n\}) = \alpha$, such that for some $s, s' \in [0,1] \cap \mathbb{Q}$ we have $S(\{1,\ldots,n\}, \leqslant, \kappa, \gamma_s, \gamma_{s'}) \neq 0$ and $\bar\delta_{\mathrm{ss}}^{\kappa(i)}(\gamma_s) \neq 0$ for $i = 1,\ldots,n$.

Let U be the finite set of all such pairs (n,κ). Let V be the finite set of elements of $C(\mathcal{A})$ of the form $\kappa(i)$ or $\kappa(\{1,\ldots,i\})$ or $\kappa(\{i,\ldots,n\})$ for $(n,\kappa) \in U$ and $1 \leqslant i \leqslant n$. Let W be the set of $w \in [0,1]$ such that for some $v, v' \in V$ we have $\gamma_w(v) = \gamma_w(v')$ but $\gamma_s(v) \neq \gamma_s(v')$ for generic $s \in [0,1]$. In fact $\gamma_w(v) = \gamma_w(v')$ is equivalent to $aw + b = 0$ for $a, b \in \mathbb{Z}$, so there is at most one w for each pair v, v', which is rational. Therefore W is a finite subset of $[0,1] \cap \mathbb{Q}$.

The point of this is that in (118) with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde\gamma$ only (n,κ) in U can give nonzero terms, and the $S(\{1,\ldots,n\}, \leqslant, \kappa, \gamma_s, \gamma_{s'})$ depend only on whether or not $\gamma_s(v) \leqslant \gamma_s(v')$ and $\gamma_{s'}(v) \leqslant \gamma_{s'}(v')$ hold for pairs v, v' in V, by Definition 4.2. But these inequalities only change when s or s' pass through a point of W. Thus, we have a finite set of 'walls' W such that $S(\{1,\ldots,n\}, \leqslant, \kappa, \gamma_s, \gamma_{s'})$ is locally constant for $s, s' \in [0,1] \setminus W$, for all n, κ that could contribute nonzero terms in (118).

Order the elements of $W \cup \{0,1\}$ as $0 = w_0 < w_1 < \cdots < w_{k-1} < w_k = 1$. We divide the remainder of the proof into three steps:

Step 1. Show (118) holds with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde\gamma$ when $s' = w_i$ and $s \in (w_{i-1}, w_i) \cap \mathbb{Q}$ or $s \in (w_i, w_{i+1}) \cap \mathbb{Q}$;

Step 2. Show (118) holds with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde\gamma$ when $s = w_i$ and $s' \in (w_{i-1}, w_i) \cap \mathbb{Q}$ or $s' \in (w_i, w_{i+1}) \cap \mathbb{Q}$;

Step 3. Show (118) holds with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde\gamma$ for any $s, s' \in [0,1] \cap \mathbb{Q}$.

Step 1. Since $s' \in W$ and no elements of W lie between s and s', we see by definition of W that $\gamma_s(v) \leqslant \gamma_s(v')$ implies $\gamma_{s'}(v) \leqslant \gamma_{s'}(v')$ for all pairs v, v' in V. If $(n,\kappa) \in U$ we see using the proof of (55) that $S(\{1,\ldots,n\}, \leqslant, \kappa, \gamma_s, \gamma_{s'})$ is 1 if $\gamma_{s'} \circ \kappa \equiv \gamma_{s'}(\alpha)$ and $\gamma_s \circ \kappa(1) > \cdots > \gamma_s \circ \kappa(n)$ and zero otherwise. Applying $\Upsilon' \circ \Pi_*$ to (45) with $\gamma_s, \gamma_{s'}$ in place of $\tau, \tilde\tau$ and using Proposition 6.20 then gives (118) with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde\gamma$.

Step 2. For s, s' as in Step 2, in Step 1 we proved (118) with $\gamma_{s'}, \gamma_s$ in place of $\gamma, \tilde{\gamma}$. The only term on the right with $n = 1$ is $I_{\rm ss}^\alpha(\gamma_{s'})^\wedge$, so we may rewrite it as

$$I_{\rm ss}^\alpha(\gamma_{s'})^\wedge = I_{\rm ss}^\alpha(\gamma_s)^\wedge - \sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ n\geqslant 2,\ \kappa(\{1,\ldots,n\})=\alpha}} S(\{1,\ldots,n\},\leqslant,\kappa,\gamma_{s'},\gamma_s)\cdot \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))}\cdot \prod_{i=1}^n I_{\rm ss}^{\kappa(i)}(\gamma_{s'})^\wedge.$$

For n, κ on the right hand side we have Hilbert polynomials $p_{\kappa(i)} = a_d^i t^d/d! + \cdots + a_0^i$ for $i = 1, \ldots, n$ which sum to p_α, so $\sum_{i=1}^n a_d^i = a_d = r + 1$ by choice of α. As $n \geqslant 2$ and $a_d^i \geqslant 1$ we have $0 < a_d^i \leqslant r$ so the inductive hypothesis $(*_r)$ applies with $\kappa(i)$ in place of α, yielding expressions for $I_{\rm ss}^{\kappa(i)}(\gamma_{s'})^\wedge$ in terms of the $I_{\rm ss}^\beta(\gamma_s)^\wedge$. Substituting these into the right hand side above gives an expression for $I_{\rm ss}^\alpha(\gamma_{s'})^\wedge$ in terms of the $I_{\rm ss}^\beta(\gamma_s)^\wedge$ and powers of ℓ. But this is exactly (118) with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde{\gamma}$, since (118) with $\gamma, \tilde{\gamma}$ exchanged is the combinatorial inverse of (118) by properties of the transformation coefficients $S(\cdots)$.

Step 3. Let $s, s' \in [0, 1] \cap \mathbb{Q}$, and suppose for simplicity that $s' < s$. Then there exist unique $1 \leqslant i \leqslant j \leqslant k$ with $w_{i-1} \leqslant s' < w_i$ and $w_{j-1} < s \leqslant w_j$. Choose $x_i, \ldots, x_j$ in $\bigl([0, 1] \cap \mathbb{Q}\bigr) \setminus W$ with $s' \leqslant x_i < w_i < x_{i+1} < w_{i+2} < \cdots < w_{j-1} < x_j \leqslant s$, where we take $x_i = s'$ if $w_{i-1} < s'$ and $x_i > s'$ if $w_{i-1} = s'$, and $x_j = s$ if $s < w_j$ and $x_j < s$ if $s = w_j$.

Then we write $I_{\rm ss}^\alpha(\gamma_{s'})^\wedge$ in terms of the $I_{\rm ss}^\beta(\gamma_s)^\wedge$ by $2(j-i+1)$ substitutions, as follows. For the first substitution, if $x_i < s'$ then $s' = w_{i-1}$, and Step 2 writes $I_{\rm ss}^\alpha(\gamma_{s'})^\wedge$ in terms of the $I_{\rm ss}^\beta(\gamma_{x_i})$, where either $\beta = \alpha$ or $p_\beta(t) = b_d t^d/d! + \cdots + b_0$ with $0 < b_d \leqslant r$, so that $(*_r)$ applies with β in place of α. If $x_i = s'$ then $I_{\rm ss}^\alpha(\gamma_{s'})^\wedge = I_{\rm ss}^\alpha(\gamma_{x_i})^\wedge$ and the first substitution is trivial.

For substitution number $2k$ for $k = 1, \ldots, j - i$ we have already written $I_{\rm ss}^\alpha(\gamma_{s'})^\wedge$ in terms of $I_{\rm ss}^\beta(\gamma_{x_{i+k-1}})^\wedge$ for $\beta = \alpha$ or β to which $(*_r)$ applies. We then use Step 1 for $\beta = \alpha$ and $(*_r)$ otherwise to write $I_{\rm ss}^\beta(\gamma_{x_{i+k-1}})^\wedge$ in terms of the $I_{\rm ss}^{\beta'}(\gamma_{w_{i+k-1}})^\wedge$ for $\beta' = \alpha$ or β' to which $(*_r)$ applies, and substitute this into the previous expression to write $I_{\rm ss}^\alpha(\gamma_{s'})^\wedge$ in terms of these $I_{\rm ss}^{\beta'}(\gamma_{w_{i+k-1}})^\wedge$. For substitution number $2k + 1$ for $k = 1, \ldots, j - i$ we use Step 2 for $\beta' = \alpha$ and $(*_r)$ otherwise to write these $I_{\rm ss}^{\beta'}(\gamma_{w_{i+k-1}})^\wedge$ in terms of $I_{\rm ss}^{\beta''}(\gamma_{x_{i+k}})^\wedge$ for $\beta'' = \alpha$ or β'' to which $(*_r)$ applies. Finally, for substitution number $2(j-i+1)$ we use Step 1 and $(*_r)$ if $x_j < s$ and do nothing if $x_j = s$.

As all we are doing at each stage is substituting in finitely many copies of (118) with different values for $\alpha, \gamma, \tilde{\gamma}$, by repeated use of (33) we see that the expressions we get for $I_{\rm ss}^\alpha(\gamma_{s'})^\wedge$ in terms of $I_{\rm ss}^\beta(\gamma_t)^\wedge$ for $t = x_i, w_i, x_{i+1}, \ldots, x_j, s$ respectively are just (118) with $\gamma_t, \gamma_{s'}$ in place of $\gamma, \tilde{\gamma}$. So at the last substitution we prove (118) with $\gamma_s, \gamma_{s'}$ in place of $\gamma, \tilde{\gamma}$, as we want. The case $s < s'$ is similar, and $s = s'$ is trivial.

In Step 3 we have proved $(*_{r+1})$ supposing $(*_r)$. Thus by induction $(*_r)$ holds for all $r \geqslant 1$. The theorem follows by setting $s = 0$ and $s' = 1$. □

An interpretation of the theorem will be proposed after Problem 7.3 below. When P is a ruled surface with K_P^{-1} nef, Yoshioka [46, Cor. 3.3] proves a result

related to Theorem 6.21, a wall-crossing formula for Poincaré polynomials of moduli spaces $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\gamma)$. It is valid only when $\gamma,\tilde\gamma$ are in the interiors of chambers separated by a single wall, in contrast to (118) which holds for arbitrary $\gamma,\tilde\gamma$.

We now define invariants $\bar J^{\alpha}(\gamma)^{\Lambda}$ related to the $J^{\alpha}(\gamma)^{\Lambda^\circ}$ of (92).

Definition 6.22. In the situation above, motivated by (94) define invariants $\bar J^{\alpha}(\gamma)^{\Lambda}\in\Lambda$ for all $\alpha\in C(\mathcal{A})$ by

$$\bar J^{\alpha}(\gamma)^{\Lambda}=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \gamma\circ\kappa\equiv\gamma(\alpha)}}\ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))}\frac{(-1)^{n-1}(\ell-1)}{n}\prod_{i=1}^{n}I_{\mathrm{ss}}^{\kappa(i)}(\gamma)^{\Lambda}. \tag{119}$$

Using the proof of [31, Th.s 6.4 & 7.7] in the algebra $A(\mathcal{A},\Lambda,\chi)$ rather than $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ we deduce an analogue of (95):

$$I_{\mathrm{ss}}^{\alpha}(\tau)^{\Lambda}=\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha,\ \tau\circ\kappa\equiv\tau(\alpha)}}\ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))}\frac{(\ell-1)^{-n}}{n!}\prod_{i=1}^{n}\bar J^{\kappa(i)}(\tau)^{\Lambda}. \tag{120}$$

There are finitely many nonzero terms in both equations, as for (21) and (23).

If (89) held in $\mathcal{A}=\mathrm{coh}(P)$ then equation (94) of Theorem 6.8 would give $\bar J^{\alpha}(\gamma)^{\Lambda}=J^{\alpha}(\gamma)^{\Lambda^\circ}$, so that $\bar J^{\alpha}(\gamma)^{\Lambda}\in\Lambda^\circ$. We can prove this in a special case, but as (89) does not hold, in general it is likely that $J^{\alpha}(\gamma)^{\Lambda^\circ}\neq\bar J^{\alpha}(\gamma)^{\Lambda}\notin\Lambda^\circ$. Thus we do not define invariants $\bar J^{\alpha}(\gamma)^{\Omega}$, as projecting to Ω may not be possible.

Proposition 6.23. *In the situation above, suppose K_P^{-1} is ample and $\alpha\in C(\mathrm{coh}(P))$ with $\dim\alpha>0$. Then $\bar J^{\alpha}(\gamma)^{\Lambda}=J^{\alpha}(\gamma)^{\Lambda^\circ}$ and lies in Λ°.*

Proof. If K_P^{-1} is ample and $0\not\cong W\in\mathrm{coh}(P)$ with $\dim W>0$, so that $\mathrm{rk}(W)\neq0$ or $c_1(W)\neq0$, the proof of Lemma 6.18 also shows that $\gamma([W])<\gamma([W\otimes K_P^{-1}])$ in G_2. Then we can modify Proposition 6.19 to show that if $X,Y\in\mathrm{coh}(P)$ are γ-semistable with $\dim X,\dim Y>0$ and $\gamma([X])=\gamma([Y])$ then $\mathrm{Ext}^2(X,Y)=0$, and Proposition 6.20 to show that if $(\{1,\ldots,n\},\leqslant,\kappa)$ is $\mathcal{A}$-data with $\kappa(\{1,\ldots,n\})=\alpha$ and $\gamma\circ\kappa\equiv\gamma(\alpha)$ and $\dim\kappa(i)>0$ then (117) holds. So applying $\Upsilon'\circ\Pi_*$ to (21) gives (94), and the proposition follows. $\square$

Combining (118)–(120) and the definition (31) of the coefficients $U(\cdots)$ we see that the $\bar J^{\alpha}(\gamma)^{\Lambda}$ transform according to (98) under change of Gieseker stability condition. Then the proof of Corollary 6.9 shows that $\bar J^{\alpha}(\gamma)^{\Lambda}$ is independent of γ if χ is symmetric. Now using Chern classes and the Riemann–Roch theorem as in Hartshorne [22, App. A] we find that for all $X,Y\in\mathcal{A}$ we have

$$\chi\bigl([X],[Y]\bigr)-\chi\bigl([Y],[X]\bigr)=\bigl(\mathrm{rk}(X)c_1(Y)-\mathrm{rk}(Y)c_1(X)\bigr)c_1(K_P).$$

Thus χ is symmetric if $c_1(K_P)=0$ in $H^2(P,\mathbb{Q})$ or $H^2(P,\mathbb{Q}_l)$, though not necessarily in $H^2(P,\mathbb{Z})$. This holds if K_P^n is trivial for some $n>0$. So from the classification of surfaces [24] we deduce:

Theorem 6.24. *In the situation of Theorem 6.21, for all $\alpha \in C(\mathcal{A})$ we have*

$$\bar{J}^\alpha(\tilde{\gamma})^\Lambda = \sum_{\substack{\mathcal{A}\text{-data } (\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \begin{aligned}[t] &U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde{\tau})\cdot \ell^{-\sum_{1\leqslant i<j\leqslant n}\chi(\kappa(j),\kappa(i))}\cdot\\ &(\ell-1)^{1-n}\textstyle\prod_{i=1}^n \bar{J}^{\kappa(i)}(\gamma)^\Lambda,\end{aligned} \tag{121}$$

with only finitely many nonzero terms. If $c_1(K_P)=0$ then the $\bar{J}^\alpha(\gamma)^\Lambda$ are independent of the choice of Gieseker stability condition $(\gamma,G_2,\leqslant)$ on $\mathcal{A}=$ $\mathrm{coh}(P)$*, that is, independent of the ample line bundle E. This holds if P is a $K3$ surface, an Enriques surface, an abelian surface, or a bielliptic surface.*

Yoshioka [46, Rem. 3.2] proves a related result, that if P is a $K3$ or abelian surface for which the Bogomolov–Gieseker inequality holds and $(\gamma,G_2,\leqslant)$ is a suitably generic Gieseker stability condition on $\mathrm{coh}(P)$, certain weighted counts of γ-semistable sheaves in class α over finite fields are independent of γ.

We now propose a conjectural means to compute many of the invariants $\bar{J}^\alpha(\gamma)^\Lambda$ when $c_1(K_P)=0$, motivated by ideas of Bridgeland [9–11]. It involves the extension of our whole programme to *triangulated categories*, and the *bounded derived category* $D^b(\mathrm{coh}(P))$ of coherent sheaves on P. This extension will be discussed in §7. Suppose for the moment that:

- There is a good notion of permissible stability condition τ on $D^b(\mathrm{coh}(P))$, based on Bridgeland stability [9], which includes the extension of Gieseker stability on $\mathrm{coh}(P)$ to $D^b(\mathrm{coh}(P))$, perhaps as a limit. These stability conditions form a moduli space, with a topology. Write $\mathrm{Stab}(P)$ for the connected component of the moduli space including Gieseker stability.
- One can define invariants $\hat{J}^\alpha(\tau)^\Lambda \in \Lambda$ 'counting' τ-semistable objects in class $\alpha \in K(\mathcal{A})$. They transform according to a generalization of (121) under change of stability conditions, at least for 'nearby' stability conditions. When τ is the extension of γ to $D^b(\mathrm{coh}(P))$ and $\alpha \in C(\mathcal{A})$ we have $\hat{J}^\alpha(\tau)^\Lambda = \bar{J}^\alpha(\gamma)^\Lambda$.
- When $c_1(K_P)=0$ the $\hat{J}^\alpha(\tau)^\Lambda$ are independent of choice of τ in the connected component $\mathrm{Stab}(P)$.
- The whole framework is preserved by autoequivalences $\Phi: D^b(\mathrm{coh}(P)) \to D^b(\mathrm{coh}(P))$ of the derived category. Write $\mathrm{Aut}^+\bigl(D^b(\mathrm{coh}(P))\bigr)$ for the group of autoequivalences preserving the connected component $\mathrm{Stab}(P)$.

Let $c_1(P)=0$, $\tau \in \mathrm{Stab}(P)$ and $\Phi \in \mathrm{Aut}^+\bigl(D^b(\mathrm{coh}(P))\bigr)$. Then for α in $K(\mathcal{A})$ we have $\hat{J}^\alpha(\tau)^\Lambda = \hat{J}^{\Phi_*(\alpha)}(\Phi_*(\tau))^\Lambda = \hat{J}^{\Phi_*(\alpha)}(\tau)^\Lambda$, so the $\hat{J}^\alpha(\tau)^\Lambda$ are unchanged by the action of Φ_* on $K(\mathcal{A})$. This suggests the following:

Conjecture 6.25. *Let Assumption 2.10 hold and P be a smooth projective surface over $\mathbb{K}$ with $c_1(K_P)=0$, and define $\mathcal{A}=\mathrm{coh}(P), K(\mathcal{A})$ and invariants $\bar{J}^\alpha(\gamma)^\Lambda$ as above. Then there exist $\hat{J}^\alpha \in \Lambda$ for $\alpha \in K(\mathcal{A})$ satisfying:*

(a) *If $\alpha \in C(\mathcal{A})$ then $\bar{J}^\alpha(\gamma)^\Lambda = \hat{J}^\alpha$.*

(b) *If $\Phi \in \mathrm{Aut}^+\bigl(D^b(\mathrm{coh}(P))\bigr)$ then $\hat{J}^{\Phi_*(\alpha)} = \hat{J}^\alpha$ for all $\alpha \in K(\mathcal{A})$.*

The conjecture could be applied in the following way. We first compute the invariants $\bar{J}^\alpha(\gamma)^\Lambda$ for some small subset S of $\alpha \in C(\mathcal{A})$ for which the moduli spaces $\mathrm{Obj}^\alpha_{\mathrm{ss}}(\gamma)$ can be explicitly understood. For example, if $\mathrm{rk}(\alpha) = 1$ and $\chi(\alpha,\alpha) = \chi(\mathcal{O},\mathcal{O}) - n$ for $n \in \mathbb{Z}$ one can show that $\mathrm{Obj}^\alpha_{\mathrm{ss}}(\gamma)$ is empty if $n < 0$ and otherwise is 1-isomorphic to $\mathrm{Jac}(P)\times\mathrm{Hilb}^n(P)\times[\mathrm{Spec}\,\mathbb{K}/\mathbb{K}^\times]$, where $\mathrm{Jac}(P)$ is the *Jacobian variety* of line bundles L on P with $[L] = [\mathcal{O}] \in K(\mathcal{A})$, and $\mathrm{Hilb}^n(P)$ is the *Hilbert scheme* of n points on P. So $\bar{J}^\alpha(\gamma)^\Lambda = 0$ for $n < 0$ and

$$\bar{J}^\alpha(\gamma)^\Lambda = (\ell-1)\Upsilon'\bigl([\mathrm{Obj}^\alpha_{\mathrm{ss}}(\gamma)]\bigr) = \Upsilon\bigl([\mathrm{Jac}(P)]\bigr)\Upsilon\bigl([\mathrm{Hilb}^n(P)]\bigr) \quad \text{for } n \geqslant 0.$$

This gives $\hat{J}^\alpha$ for $\alpha \in S$, so (b) determines $\hat{J}^\alpha$ for $\alpha \in \mathrm{Aut}^+\bigl(D^b(\mathrm{coh}(P))\bigr)\cdot S$, and then (a) gives $\bar{J}^\alpha(\gamma)^\Lambda$ for $\alpha \in \bigl(\mathrm{Aut}^+(D^b(\mathrm{coh}(P)))\cdot S\bigr) \cap C(\mathcal{A})$. Thus, provided we understand the action of $\mathrm{Aut}^+\bigl(D^b(\mathrm{coh}(P))\bigr)$ on $K(\mathcal{A})$ reasonably well, we can compute the invariants $\bar{J}^\alpha(\gamma)^\Lambda$ on a much larger subset of $C(\mathcal{A})$, perhaps even the whole of it.

Let P be an algebraic $K3$ surface over $\mathbb{C}$. Using his notion of stability condition on triangulated categories [9], Bridgeland [10] (surveyed in [11, §6]) parametrizes a connected component $\mathrm{Stab}(P)$ of the moduli space of stability conditions on $D^b(\mathrm{coh}(P))$. Bridgeland's definition does not include the extension of Gieseker stability on $\mathrm{coh}(P)$ to $D^b(\mathrm{coh}(P))$, but can probably be generalized so that it does, perhaps following Gorodentscev et al. [19].

Using results of Orlov [37] on autoequivalences of $D^b(\mathrm{coh}(P))$, Bridgeland [10, Conj. 1.2] conjecturally describes $\mathrm{Aut}^+(D^b(\mathrm{coh}(P)))$. His description implies that $\mathrm{Aut}^+(D^b(\mathrm{coh}(P)))$ acts on $H^*(P,\mathbb{Z})$ as a certain index 2 subgroup of the group of automorphisms of $H^*(P,\mathbb{Z})$ preserving the Mukai form and the subspace $H^{2,0}(P) \subset H^2(P,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}$. Combining this with Conjecture 6.25 should make an effective tool for conjecturally computing invariants $\bar{J}^\alpha(\gamma)^\Lambda$ when P is a complex algebraic $K3$ surface, which might lead to new formulae for Poincaré and Hodge polynomials for moduli spaces of γ-semistable sheaves on P.

As evidence for Conjecture 6.25 we note some results of Yoshioka [47]. Let P be an abelian surface, $\alpha \in C(\mathcal{A})$ *primitive* with $\chi(\alpha,\alpha) \leqslant -2$ and $(\gamma, G_2, \leqslant)$ a Gieseker stability condition with $\mathrm{Obj}^\alpha_{\mathrm{st}}(\gamma) = \mathrm{Obj}^\alpha_{\mathrm{ss}}(\gamma)$, so that $\mathrm{Obj}^\alpha_{\mathrm{ss}}(\gamma) \cong \mathcal{M}^\alpha_{\mathrm{ss}}(\gamma) \times [\mathrm{Spec}\,\mathbb{K}/\mathbb{K}^\times]$ for a nonsingular projective symplectic variety $\mathcal{M}^\alpha_{\mathrm{ss}}(\gamma)$ of dimension $2 - \chi(\alpha,\alpha)$. Then [47, Th. 0.1] shows $\mathcal{M}^\alpha_{\mathrm{ss}}(\gamma)$ is deformation equivalent to $\mathrm{Jac}(P) \times \mathrm{Hilb}^{-\chi(\alpha,\alpha)/2}(P)$. If instead P is a $K3$ surface then [47, Th. 8.1] shows $\mathcal{M}^\alpha_{\mathrm{ss}}(\gamma)$ is deformation equivalent to $\mathrm{Hilb}^{1-\chi(\alpha,\alpha)/2}(P)$.

Now if our choice of Υ in Assumption 2.10 is unchanged by deformations of smooth projective $\mathbb{K}$-varieties, which holds for virtual Poincaré polynomials and virtual Hodge polynomials when $\mathbb{K} = \mathbb{C}$, these results allow us to evaluate $\bar{J}^\alpha(\gamma)^\Lambda$ for most primitive $\alpha \in C(\mathcal{A})$. The answer depends only on P and $\chi(\alpha,\alpha)$, and so is invariant under $\mathrm{Aut}^+\bigl(D^b(\mathrm{coh}(P))\bigr)$ as in Conjecture 6.25(b).

We quote Bridgeland [11, §6]:

> 'As a final remark in this section note that Borcherd's work on modular forms [7] allows one to write down product expansions for holomorphic functions on $\mathrm{Stab}(P)$ that are invariant under the group

$\mathrm{Aut}(D^b(\mathrm{coh}(P)))$. It would be interesting to connect these formulae with counting invariants for stable objects in $D^b(\mathrm{coh}(P))$.'

If Conjecture 6.25 and the reasoning behind it are correct, then the $\hat{J}^\alpha$ we propose are invariants 'counting' τ-semistable objects in class α, which are independent of choice of τ. Following Bridgeland's suggestion, the author wonders if the $\hat{J}^\alpha$ can be combined in a generating function on $\mathrm{Stab}(P)$ to give one of Borcherd's automorphic forms of weight $k \geqslant 1$. One might try something like:

$$f(\tau) = \textstyle\sum_{\alpha \in K(\mathcal{A}) \setminus \{0\}} \hat{J}^\alpha Z(\alpha)^{-k}, \tag{122}$$

where $Z : K(\mathcal{A}) \to \mathbb{C}$ is the 'central charge' associated to τ. If the sum converges absolutely then $f : \mathrm{Stab}(P) \to \Lambda \otimes_{\mathbb{Q}} \mathbb{C}$ is a holomorphic function. See [32] for results on how to encode the invariants of this paper into holomorphic generating functions on complex manifolds of stability conditions.

Donaldson invariants are invariants of smooth 4-manifolds P described in Donaldson and Kronheimer [16], which when P is a Kähler surface effectively 'count' stable rank 2 holomorphic vector bundles in class α on P. When $b^2_+(P) > 1$ these invariants depend only on P as a smooth 4-manifold, and so are unchanged under deformations of P. When $b^2_+(P) = 1$ they depend on a little more: they are defined using a metric g on P, which determines a splitting $H^2(P, \mathbb{R}) = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$, and the invariants depend on this splitting and have wall-crossing behaviour when $\mathcal{H}^2_- \cap H^2(P, \mathbb{Z}) \neq \{0\}$.

If K_P^{-1} is nef but not trivial then $H^{2,0}(P) = 0$, so $b^2_+(P) = 1$, and $\mathcal{H}^2_+ = \langle [\omega] \rangle$ is spanned by the cohomology class of the Kähler form ω. This depends on the ample line bundle E used to embed P into projective space, and so on the stability condition $(\gamma, G_2, \leqslant)$. Thus, the wall-crossing behaviour for Donaldson invariants when $b^2_+(P) = 1$ is directly analogous to the transformation laws (118), (121) for our invariants under change of $(\gamma, G_2, \leqslant)$. When K_P is trivial we have $b^2_+(P) = 3$, so Donaldson invariants are independent of $\mathcal{H}^2_\pm$; this is analogous to $\bar{J}^\alpha(\gamma)^\Lambda$ being independent of γ in this case.

The author wonders whether there exist invariants related to Donaldson invariants and similar to $I^\alpha_{\mathrm{ss}}(\gamma)^\Lambda, \bar{J}^\alpha(\gamma)^\Lambda$ above, for which the analogues of Theorems 6.21 and 6.24 hold, but which are *independent of deformations of P which do not change $K(\mathrm{coh}(P))$*. When $\mathbb{K} = \mathbb{C}$ this means not changing $H^{1,1}(P) \cap H^2(P, \mathbb{Q})$ in $H^2(P, \mathbb{C})$. We discuss a similar question for Calabi–Yau 3-folds and Donaldson–Thomas invariants in Conjecture 6.30 below.

6.5 Invariants of Calabi–Yau 3-folds

We now prove that when $\mathcal{A} = \mathrm{coh}(P)$ for P a Calabi–Yau 3-fold the invariants $J^\alpha(\tau)^\Omega, J^{\mathrm{b}}_{\mathrm{si}}(I, \preceq, \kappa, \tau)^\Omega$ of (93) have special properties. Suppose Assumptions 2.10 and 3.5 hold, and $\bar{\chi} : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ is biadditive and satisfies

$$\begin{aligned} &\bigl(\dim_{\mathbb{K}} \mathrm{Hom}(X, Y) - \dim_{\mathbb{K}} \mathrm{Ext}^1(X, Y)\bigr) - \\ &\bigl(\dim_{\mathbb{K}} \mathrm{Hom}(Y, X) - \dim_{\mathbb{K}} \mathrm{Ext}^1(Y, X)\bigr) = \bar{\chi}\bigl([X], [Y]\bigr) \quad \text{for all } X, Y \in \mathcal{A}. \end{aligned} \tag{123}$$

We showed in [30, §6.6] using Serre duality that this holds if $\mathcal{A} = \mathrm{coh}(P)$ for P a Calabi–Yau 3-fold. Also, (89) implies (123) with $\bar\chi(\alpha,\beta) = \chi(\alpha,\beta) - \chi(\beta,\alpha)$, so (123) is a weakening of (89), and thus holds for $\mathcal{A} = \mathrm{coh}(P)$ with P a smooth projective curve and for $\mathcal{A} = \text{mod-}\mathbb{K}Q$, as in §6.2.

Under these assumptions, in [30, §6.6] we defined a *Lie algebra morphism*

$$\Psi^\Omega \circ \bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}} : \mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}}) \to C^{\mathrm{ind}}(\mathcal{A},\Omega,\tfrac{1}{2}\bar\chi), \tag{124}$$

where $C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$ is an explicit Lie algebra contained in an explicit algebra $C(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$. Now $C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$ is an Ω-module with Ω-basis c^α for $\alpha \in C(\mathcal{A})$, and comparing Definition 6.7 with [30, Def. 6.10] shows that

$$\begin{aligned} \Psi^\Omega \circ \bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\bar\delta^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau)\bigr) &= J^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau)^\Omega\, c^{\kappa(I)} \\ \text{and} \quad \Psi^\Omega \circ \bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\bar\epsilon^\alpha(\tau)\bigr) &= J^\alpha(\tau)^\Omega\, c^\alpha. \end{aligned} \tag{125}$$

We use this to prove *multiplicative identities* on the $J^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau)^\Omega$.

Theorem 6.26. *Let Assumptions 2.10 and 3.5 hold, $\bar\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ be biadditive and satisfy* (123)*, and $(\tau,T,\leqslant)$ be a permissible weak stability condition on $\mathcal{A}$. Suppose $(I,\preceq,\kappa),(J,\lesssim,\lambda)$ are $\mathcal{A}$-data with $(I,\preceq),(J,\lesssim)$ connected and $I\cap J=\emptyset$. Define $K=I\amalg J$ and $\mu : K\to C(\mathcal{A})$ by $\mu|_I=\kappa$ and $\mu|_J=\lambda$. Then*

$$\begin{aligned} &\sum_{\substack{\text{p.o.s } \trianglelefteq \text{ on } K:\ (K,\trianglelefteq) \text{ connected,} \\ \trianglelefteq|_I = \preceq,\ \trianglelefteq|_J = \lesssim, \text{ and} \\ i\in I,\ j\in J \text{ implies } j \ntrianglelefteq i}} J^{\mathrm{b}}_{\mathrm{si}}(K,\trianglelefteq,\mu,\tau)^\Omega - \sum_{\substack{\text{p.o.s } \trianglelefteq \text{ on } K:\ (K,\trianglelefteq) \text{ connected,} \\ \trianglelefteq|_I = \preceq,\ \trianglelefteq|_J = \lesssim, \text{ and} \\ i\in I,\ j\in J \text{ implies } i \ntrianglelefteq j}} J^{\mathrm{b}}_{\mathrm{si}}(K,\trianglelefteq,\mu,\tau)^\Omega \\ &\qquad = \bar\chi\bigl(\kappa(I),\lambda(J)\bigr) J^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau)^\Omega J^{\mathrm{b}}_{\mathrm{si}}(J,\lesssim,\lambda,\tau)^\Omega. \end{aligned} \tag{126}$$

Proof. As (124) is a Lie algebra morphism, by (125) we have

$$\begin{aligned} &\Psi^\Omega \circ \bar\Pi^{\Theta,\Omega}_{\mathfrak{Obj}_{\mathcal{A}}}\bigl(\bigl[\bar\delta^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau),\bar\delta^{\mathrm{b}}_{\mathrm{si}}(J,\lesssim,\lambda,\tau)\bigr]\bigr) = \\ &\qquad\qquad J^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau)^\Omega J^{\mathrm{b}}_{\mathrm{si}}(J,\lesssim,\lambda,\tau)^\Omega [c^{\kappa(I)},c^{\lambda(J)}]. \end{aligned}$$

Also $[c^{\kappa(I)},c^{\lambda(J)}] = \bar\chi\bigl(\kappa(I),\lambda(J)\bigr)c^{\kappa(I)+\lambda(J)}$ by definition of $C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$. The multiplicative relations between the $\bar\delta^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau)$ in $\bar{\mathcal{H}}^{\mathrm{pa}}_\tau$ are given explicitly in [31, §7.1 & §8]. Using these we can write $\bigl[\bar\delta^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau),\bar\delta^{\mathrm{b}}_{\mathrm{si}}(J,\lesssim,\lambda,\tau)\bigr]$ as a linear combination of $\bar\delta^{\mathrm{b}}_{\mathrm{si}}(K,\trianglelefteq,\mu,\tau)$ over different $\trianglelefteq$. Combining all this and (125) gives (126), but without the conditions $(K,\trianglelefteq)$ connected. However, as $(I,\preceq),(J,\lesssim)$ are connected the only possibility for $\trianglelefteq$ with $(K,\trianglelefteq)$ disconnected is given by $a \trianglelefteq b$ if and only if $a,b\in I$ with $a\preceq b$ or $a,b\in J$ with $a\lesssim b$. As this appears in both sums with opposite signs, the terms in disconnected $(K,\trianglelefteq)$ cancel out, so we may restrict to connected $(K,\trianglelefteq)$ in (126). $\square$

Next we discuss how the invariants depend on $(\tau,T,\leqslant)$. The $J^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau)^{\Lambda^\circ}$ and $J^{\mathrm{b}}_{\mathrm{si}}(I,\preceq,\kappa,\tau)^\Omega$ satisfy *additive* transformation laws similar to (82), which may be deduced from (82) and identities in [31, §8]. The $J^\alpha(\tau)^\Omega$ satisfy a

multiplicative transformation law, (129) below. To deduce it we need some facts about multiplication in $C(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$.

Suppose $\alpha_1,\ldots,\alpha_n \in C(\mathcal{A})$, so that $c^{\alpha_1},\ldots,c^{\alpha_n} \in C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$. Then using the multiplication relations in [30, Def. 6.10] we can compute $c^{\alpha_1}\star c^{\alpha_2}\star\cdots\star c^{\alpha_n}$ in the algebra $C(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$. This is rather complicated, but we will only need to know the component in $C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar\chi)$, that is, the coefficient of $c^{\alpha_1+\cdots+\alpha_n}$ in this sum. Calculation shows this is given by:

$$
\begin{aligned}
c^{\alpha_1}\star\cdots\star c^{\alpha_n} = {} & \text{ terms in } c_{[I,\kappa]},\ |I|>1, \\
+ & \left[\frac{1}{2^{n-1}}\sum_{\substack{\text{connected, simply-connected digraphs }\Gamma:\\ \text{vertices }\{1,\ldots,n\},\text{ edge } \overset{i}{\bullet}\rightarrow\overset{j}{\bullet} \text{ implies } i<j}}\ \prod_{\substack{\text{edges}\\ \overset{i}{\bullet}\rightarrow\overset{j}{\bullet}\\ \text{in }\Gamma}}\bar\chi(\alpha_i,\alpha_j)\right]c^{\alpha_1+\cdots+\alpha_n}.
\end{aligned}
\tag{127}
$$

Here a *digraph* is a directed graph.

Since (49) can be regarded as an identity in the Lie algebra $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ by Theorem 5.4, we can apply the Lie algebra morphism (124) to (49) to get an identity writing $J^\alpha(\tilde\tau)^\Omega$ in terms of the $J^{\kappa(i)}(\tau)^\Omega$. This is expressed as a sum over all $\mathcal{A}$-data $(\{1,\ldots,n\},\leqslant,\kappa)$ followed by a sum over digraphs Γ with vertices $\{1,\ldots,n\}$. To simplify this formula we define some new notation.

Definition 6.27. Suppose Condition 4.1 holds for $(\tau,T,\leqslant)$ and $(\tilde\tau,\tilde T,\leqslant)$, and let Γ be a connected, simply-connected digraph with finite vertex set I, and $\kappa: I\rightarrow C(\mathcal{A})$. Define $V(I,\Gamma,\kappa,\tau,\tilde\tau)\in\mathbb{Q}$ by

$$
V(I,\Gamma,\kappa,\tau,\tilde\tau) = \frac{1}{2^{|I|-1}|I|!}\sum_{\substack{\text{total orders }\preceq\text{ on } I:\\ \text{edge } \overset{i}{\bullet}\rightarrow\overset{j}{\bullet} \text{ in } \Gamma \text{ implies } i\preceq j}} U(I,\preceq,\kappa,\tau,\tilde\tau). \tag{128}
$$

With this we prove a transformation law for the $J^\alpha(\tau)^\Omega$.

Theorem 6.28. *Let Assumptions 2.10 and 3.5 hold and $\bar\chi: K(\mathcal{A})\times K(\mathcal{A})\rightarrow\mathbb{Z}$ be biadditive and satisfy* (123). *Suppose $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ are permissible weak stability conditions on $\mathcal{A}$, the change from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ is globally finite, and there exists a weak stability condition $(\hat\tau,\hat T,\leqslant)$ on $\mathcal{A}$ dominating $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ with the change from $(\hat\tau,\hat T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$ locally finite. Then for all $\alpha\in C(\mathcal{A})$ the following holds in Ω, with only finitely many nonzero terms:*

$$
\begin{aligned}
J^\alpha(\tilde\tau)^\Omega = \sum_{\substack{\textit{iso.}\\ \textit{classes}\\ \textit{of finite}\\ \textit{sets } I}}\ \sum_{\substack{\kappa: I\rightarrow C(\mathcal{A}):\\ \kappa(I)=\alpha}}\ \sum_{\substack{\textit{connected,}\\ \textit{simply-connected}\\ \textit{digraphs }\Gamma,\\ \textit{vertices } I}} V(I,\Gamma,\kappa,\tau,\tilde\tau) & \cdot\prod_{\textit{edges } \overset{i}{\bullet}\rightarrow\overset{j}{\bullet} \textit{ in } \Gamma}\bar\chi(\kappa(i),\kappa(j)) \\
& \cdot\prod_{i\in I} J^{\kappa(i)}(\tau)^\Omega.
\end{aligned}
\tag{129}
$$

Proof. Theorem 5.2 implies that (49) holds, with only finitely many nonzero terms. Theorem 5.4 shows (49) can be rewritten as a Lie algebra identity, a

linear combination of multiple commutators in $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. So applying the Lie algebra morphism (124) yields a Lie algebra identity in $C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar{\chi})$, which by (125) writes $J^\alpha(\tilde{\tau})^\Omega c^\alpha$ as a linear combination of multiple commutators of $J^{\kappa(i)}(\tau)^\Omega c^{\kappa(i)}$. But $C^{\mathrm{ind}}(\mathcal{A},\Omega,\frac{1}{2}\bar{\chi})$ is a Lie subalgebra of the algebra $C(\mathcal{A},\Omega,\frac{1}{2}\bar{\chi})$, so we can regard this as an identity in $C(\mathcal{A},\Omega,\frac{1}{2}\bar{\chi})$, avoiding the need to rewrite (49) in terms of multiple commutators. This proves:

$$J^\alpha(\tilde{\tau})^\Omega c^\alpha = \sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde{\tau})\bigl[\textstyle\prod_{i=1}^n J^{\kappa(i)}(\tau)^\Omega\bigr]c^{\kappa(1)}\star\cdots\star c^{\kappa(n)}.$$

Equating coefficients of c^α on both sides of this equation, and noting that the coefficient in $c^{\kappa(1)}\star\cdots\star c^{\kappa(n)}$ is given by (127), yields:

$$\begin{aligned} J^\alpha(\tilde{\tau})^\Omega = &\sum_{\substack{\mathcal{A}\text{-data }(\{1,\ldots,n\},\leqslant,\kappa):\\ \kappa(\{1,\ldots,n\})=\alpha}} \;\sum_{\substack{\text{connected, simply-connected digraphs }\Gamma:\\ \text{vertices }\{1,\ldots,n\},\text{ edge }\overset{i}{\bullet}\to\overset{j}{\bullet}\text{ implies }i<j}} \\ &\frac{1}{2^{n-1}}U(\{1,\ldots,n\},\leqslant,\kappa,\tau,\tilde{\tau})\prod_{\text{edges }\overset{i}{\bullet}\to\overset{j}{\bullet}\text{ in }\Gamma}\bar{\chi}(\kappa(i),\kappa(j))\prod_{i=1}^n J^{\kappa(i)}(\tau)^\Omega. \end{aligned} \tag{130}$$

We claim that substituting (128) into (129) gives a sum equivalent to (130). For substituting (128) into (129) gives sums over I, κ, Γ, and total orders $\preceq$ on I. The relationship with (130) is that having chosen $I,\Gamma,\preceq$, there is a unique isomorphism $(I,\preceq)\cong(\{1,\ldots,n\},\leqslant)$, where $n=|I|$. So identifying I with $\{1,\ldots,n\}$ relates the sums over $\Gamma,\preceq$ in (128)–(129) to the sums over n,Γ in (130). But this is not a 1-1 correspondence: rather, to each choice of n,Γ in (130) there are $|I|!$ choices of $I,\Gamma,\preceq$ in the substitution of (128) in (129), as there are $|I|!$ total orders $\preceq$ on I. This is cancelled by the factor $1/|I|!$ in (128). All the other terms in (128)–(129) and (130) immediately agree. $\square$

Remark 6.29. The author can prove that in Definition 6.27, if $\tilde{\Gamma}$ is a directed graph obtained from Γ by reversing the directions of k edges in Γ then $V(I,\tilde{\Gamma},\kappa,\tau,\tilde{\tau})=(-1)^k V(I,\Gamma,\kappa,\tau,\tilde{\tau})$. Since $\bar{\chi}$ is antisymmetric, replacing Γ by $\tilde{\Gamma}$ also multiplies the product $\prod_{\overset{i}{\bullet}\to\overset{j}{\bullet}\text{ in }\Gamma}\bar{\chi}(\kappa(i),\kappa(j))$ in (129) by $(-1)^k$.

Thus the product of terms on the r.h.s. of (129) is actually *independent* of the orientation of Γ, and depends only on the underlying undirected graph. As there are $2^{|I|-1}$ orientations on this graph, we could omit the factor $1/2^{|I|-1}$ in (128) and write (129) as a sum over undirected graphs rather than digraphs.

Equation (129) will play an important rôle in a sequel [32]. Neglecting issues to do with convergence of infinite sums, we encode the invariants $J^\alpha(\tau)^\Omega$ in *holomorphic generating functions* on the complex manifold of stability conditions. Because the $J^\alpha(\tau)^\Omega$ satisfy the multiplicative transformation law (129), these generating functions satisfy a *nonlinear p.d.e.*, which can be interpreted as the *flatness of a connection* over the complex manifold of stability conditions.

In the remainder of the section we discuss how our results should relate to other proposed invariants of Calabi–Yau 3-folds, and the whole Mirror Symmetry picture. Motivated by Donaldson and Thomas [17, p. 33–34], Thomas

[40] defines invariants $DT^\alpha(\tau)$ 'counting' τ-stable coherent sheaves in class $\alpha \in C(\mathrm{coh}(P))$ on a Calabi–Yau 3-fold P, which are now known as *Donaldson–Thomas invariants.* We compare these with our invariants $J^\alpha(\tau)^\Omega$ above, which also 'count' τ-semistable coherent sheaves in class α on P:

- The main good property of Donaldson–Thomas invariants $DT^\alpha(\tau)$ is that they are *unchanged* under deformations of the complex structure of P. Thomas achieves this by using a *virtual moduli cycle* to cut the moduli schemes down to the expected dimension (zero) and then counting points.

  In contrast, our invariants $J^\alpha(\tau)^\Omega$ are *not* expected to be unchanged under deformations of P. This is because rather than using virtual moduli cycles we just take a motivic invariant, such as an Euler characteristic, of the moduli scheme as it stands. This is quite a crude thing to do.

- Donaldson–Thomas invariants are (so far) defined only for $\alpha \in C(\mathcal{A})$ for which $\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau) = \mathrm{Obj}_{\mathrm{st}}^\alpha(\tau)$, that is, τ-semistability and τ-stability coincide. This is because strictly τ-semistable objects would give singular points of the moduli space which the virtual moduli cycle technology is presently unable to cope with.

  In contrast, our invariants $J^\alpha(\tau)^\Omega$ are defined for all $\alpha \in C(\mathcal{A})$. Moreover, a great deal of the work in this paper and [27–31] is really about how to deal with strictly τ-semistables – for instance, $\epsilon^\alpha(\tau), \bar\epsilon^\alpha(\tau)$ coincide with $\delta_{\mathrm{ss}}^\alpha(\tau), \bar\delta_{\mathrm{ss}}^\alpha(\tau)$ except over strictly τ-semistables, and much of [27, 28, 30] concerns how best to include stabilizer groups when forming invariants of subsets of stacks, and this is not relevant for τ-stable objects as they all have stabilizer group $\mathbb{K}^\times$.

- The transformation laws for Donaldson–Thomas invariants under change of stability condition are not known. In contrast, our invariants $J^\alpha(\tau)^\Omega$ transform according to (129).

- Donaldson–Thomas invariants are defined uniquely and take values in $\mathbb{Z}$. In contrast, our invariants depend on a choice of motivic invariant Θ in Assumption 2.10, with values in a $\mathbb{Q}$-algebra Ω. As in [28, Ex. 6.3], possibilities for Θ include the Euler characteristic χ, and the sum of the virtual Betti numbers.

- Donaldson and Thomas worked hard to produce invariants whose behaviour under deformation of P is understood. In contrast, we have worked hard to produce invariants whose behaviour under change of weak stability condition $(\tau, T, \leqslant)$ is understood.

We would now like to conjecture that there exist invariants 'counting' τ-semistable coherent sheaves on P which combine the good features of both Donaldson–Thomas invariants and our own.

Conjecture 6.30. *Fix $\mathbb{K} = \mathbb{C}$, let P be a Calabi–Yau 3-fold, and define $\mathcal{A} = \mathrm{coh}(P)$, $\mathfrak{F}_{\mathcal{A}}$ and $K(\mathcal{A})$ as in* [29, §9.1]. *Let $(\tau, T, \leqslant)$ be a permissible stability condition of Gieseker type on $\mathcal{A}$, as in* [31, §4.4]. *Then there*

should exist extended Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$ defined for all $\alpha \in C(\mathcal{A})$, which should be unchanged under deformations of P, and should transform according to (129) *under change of stability condition. If $\alpha \in C(\mathcal{A})$ with $\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau) = \mathrm{Obj}_{\mathrm{st}}^\alpha(\tau)$, so that the Donaldson–Thomas invariant $DT^\alpha(\tau)$ is defined as in* [40]*, we have $\bar{DT}^\alpha(\tau) = DT^\alpha(\tau) \in \mathbb{Z}$.*

There may also exist more complicated systems of invariants analogous to the $I_{\mathrm{ss}}(I, \preceq, \kappa, \tau)$ or $J_{\mathrm{si}}^{\mathrm{b}}(I, \preceq, \kappa, \tau)^\Omega$ above, which take values in $\mathbb{Q}$, are unchanged under deformations of P, and transform in the appropriate way under change of stability condition.

I believe that proving this conjecture is feasible, although difficult, and that proving it will probably involve an extension to virtual moduli cycle technology. I have plans to attempt this in the next few years, in collaboration with others.

The invariants of Conjecture 6.30 should be defined using 0-dimensional virtual moduli cycles, which are basically finite sets of points with integer multiplicities. The only information in them independent of choices is the number of points $[X]$, counted with multiplicity and weighted by the correct function of the stabilizer group $\mathrm{Aut}(X)$, probably $\chi(\mathrm{Aut}(X)/T_X)^{-1}$, where T_X is a maximal torus in $\mathrm{Aut}(X)$. This is why we say the invariants should take values in $\mathbb{Q}$, in contrast to the rest of the section where our invariants take values in more general algebras $\Lambda, \Lambda^\circ, \Omega$.

Now under the *Homological Mirror Symmetry programme* of Kontsevich [33], (semi)stable coherent sheaves on P are supposed to be mirror to *special Lagrangian* 3*-folds* (*SL* 3*-folds*) in the mirror Calabi–Yau 3-fold M. So we expect invariants counting (semi)stable (complexes of) coherent sheaves on P to be equal to other invariants counting SL 3-folds in M, at least if these invariants count anything meaningful in String Theory.

Motivated by a detailed study [26] of the singularities of SL m-folds, in [25] I made a conjecture that there should exist interesting invariants $K^\alpha(J)$ of M that 'count' SL homology 3-spheres in a given class $\alpha \in H_3(M, \mathbb{Z})$. I expected these invariants to be independent of the choice of Kähler form on M, and to transform according to some wall-crossing formulae under deformation of complex structure J, but at the time I could only determine these transformation laws in the simplest cases. I can now expand this conjecture to specify these transformation laws.

Conjecture 6.31. *Let (M, J, ω, Ω) be an (almost) Calabi–Yau* 3*-fold, with compact* 6*-manifold M, complex structure J, Kähler form ω and holomorphic volume form Ω. Then there should exist invariants $K^\alpha(J) \in \mathbb{Q}$ for $\alpha \in H_3(M, \mathbb{Z})$ which 'count' special Lagrangian homology* 3*-spheres N in M with $[N] = \alpha$, and probably other kinds of immersed or singular SL* 3*-folds as well. These invariants $K^\alpha(J)$ are independent of the Kähler form ω and of complex rescalings of Ω, and so depend only on the complex structure J.*

The holomorphic volume form Ω determines a stability condition τ of Bridgeland type [9] *on the derived Fukaya category $D^b(F(M, \omega))$ of the symplectic manifold (M, ω), a triangulated category, and SL* 3*-folds N whose Floer homology is unobstructed correspond to τ-semistable objects in $D^b(F(M, \omega))$. Thus, the*

$K^\alpha(J)$ are invariants counting τ-semistable objects in class α in $H_3(M,\mathbb{Z}) = K(D^b(F(M,\omega)))$. Under deformation of complex structure of M, the $K^\alpha(J)$ transform according to the triangulated category extension of (129).

If P is a mirror Calabi–Yau 3*-fold to M, then the $K^\alpha(J)$ should be equal to invariants counting $\tilde\tau$-semistable objects in the derived category $D^b(\mathrm{coh}(P))$ of coherent sheaves on P, with respect to a stability condition $\tilde\tau$ on $D^b(\mathrm{coh}(P))$ of Bridgeland type. These invariants are a triangulated category version of the extended Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau)$ of Conjecture 6.30, and coincide with them if $\tilde\tau$ is induced by Gieseker stability on* $\mathrm{coh}(P)$*.*

There may also exist more complicated systems of invariants analogous to the $I_{\mathrm{ss}}(I,\preceq,\kappa,\tau)$ or $J_{\mathrm{si}}^{\mathrm{b}}(I,\preceq,\kappa,\tau)^\Omega$ above, that count configurations of SL 3*-folds, take values in $\mathbb{Q}$, are independent of ω, and transform in the appropriate way under change of complex structure. This set-up may generalize to (almost) Calabi–Yau m-folds for all $m \geqslant 2$.*

The extension of our programme to triangulated categories will be discussed at length in §7, so we will say no more about it here. I expect Conjecture 6.31 to be extremely difficult to prove, much more so than Conjecture 6.30, even just the part concerning the definition of $K^\alpha(J)$, independence of ω, and transformation laws under deformation of J. This is because we cannot use the machinery of algebraic geometry, and instead need to know a lot about the singular behaviour of SL 3-folds, which is at the moment only partially understood.

In fact clarifying Conjecture 6.31 was the beginning of this whole project, which eventually grew to [27–31] and this paper, and may continue to expand. I wanted to know more about the proposed invariants $K^\alpha(J)$, in particular their transformation laws under change of J, so I decided to study the mirror problem of counting semistable coherent sheaves on Calabi–Yau 3-folds using algebraic geometry. I didn't realize at the time how large an undertaking this would be.

The necessity of introducing, and counting, *configurations* in order to understand transformation laws under change of stability condition came directly out of my work [26, §9] on creation of new SL m-folds by multiple connected sums under deformation of the underlying Calabi–Yau m-fold.

I believe that the invariants $\bar{DT}^\alpha(\tau)$ and $K^\alpha(J)$ proposed in Conjectures 6.30 and 6.31 should play an important part in a chapter of the Mirror Symmetry story that has not yet been understood. They should encode a lot about the structure of the 'stringy Kähler moduli space', teach us about *branes* and Π-*stability* in String Theory, and perhaps have other applications.

Note that Conjecture 6.31 has predictive power for the kind of wall-crossing phenomena in special Lagrangian geometry considered in [26, §9]. For example, we can consider a smooth family of Calabi–Yau 3-folds $M_t : t \in (-\epsilon,\epsilon)$ with compact nonsingular SL homology 3-spheres N_1, N_2 in M_0 of the same phase, intersecting transversely in a single point. Then [26, Th. 9.7] gives necessary and sufficient criteria for the existence of a connected sum SL 3-fold $N_1 \# N_2$ in M_t for small $t > 0$ or $t < 0$.

Using Conjecture 6.31 and a little geometric and algebraic intuition, we can make predictions for the number of SL 3-folds in M_t of the general form

$a_1N_1 \# a_2N_2$ for $a_1, a_2 > 0$. This is not at all clear using the techniques of [26], as for $a_1 > 1$ the connected sum 'necks' might occur at any point in N_1, and we might have to consider branching or covering phenomena over N_1 too.

7 Questions for further research

We have already discussed a number of research problems in §6.3–§6.5, in particular Conjectures 6.25, 6.30 and 6.31. We now pose some more, and in the process sketch out how the author would like to see the subject develop in future. Perhaps the most important is:

Problem 7.1. *Develop an extension of the work of* [29–31] *and this paper from abelian categories to triangulated categories, and apply it to derived categories* $D^b(\mathcal{A})$ *when* $\mathcal{A}$ *is a category of quiver representations* $\text{mod-}\mathbb{K}Q, \text{mod-}\mathbb{K}Q/I$ *or coherent sheaves* $\text{coh}(P)$.

Here are some issues and difficulties involved in this:

- *Defining configurations in triangulated categories* $\mathcal{T}$. This looks straightforward: in Definition 3.2 we replace the exact sequences (8) by distinguished triangles. This involves introducing a third family of morphisms $\partial(L, J) : \sigma(L) \to \sigma(J)[1]$ in $\mathcal{T}$, so that a configuration is a quadruple $(\sigma, \iota, \pi, \partial)$, and we must add some extra axioms involving the $\partial(L, J)$. For example, a $(\{1, \ldots, n\}, \leqslant)$-configuration will be equivalent to a *distinguished* n*-dimensional hypersimplex* in $\mathcal{T}$, [18, p. 260-1].
- *Constructing moduli stacks of objects and configurations in* $\mathcal{T}$. This seems difficult, and neither triangulated categories nor Artin stacks may be the right frameworks. Toën [41–43] works instead with *dg-categories* [41], a richer structure from which one can often recover the triangulated categories we are interested in as the homotopy category. When $\mathcal{T}$ is a dg-category Toën and Vaquié [43] construct moduli ∞*-stacks* and D^-*-stacks* of objects in $\mathcal{T}$, some of which can be truncated to Artin stacks.

  For our applications we may not need the moduli 'stack' of all objects in $\mathcal{T}$, but only those which could be τ-semistable for some stability condition τ, or appear in the heart of some t-structure. So we can probably restrict to $X \in \mathcal{T}$ with $\text{Ext}^i(X, X) = 0$ for $i < 0$, which may improve the problem.
- *Non-representable morphisms.* For abelian categories $\boldsymbol{\sigma}(I) : \mathfrak{M}(I, \preceq)_{\mathcal{A}} \to \mathfrak{Obj}_{\mathcal{A}}$ is a *representable* 1-morphism, but in the triangulated category case this will no longer hold. This is because for abelian categories the automorphism group of an exact sequence injects into the automorphism group of its middle term, but the analogue fails for distinguished triangles in triangulated categories.

  Because of this, the multiplication $*$ and operations $P_{(I,\preceq)}$ on $\text{CF}(\mathfrak{Obj}_{\mathcal{A}})$, $\text{SF}(\mathfrak{Obj}_{\mathcal{A}})$ in [30] cannot be defined in the triangulated category case. We can still work with the spaces $\underline{\text{SF}}(\mathfrak{Obj}_{\mathcal{A}})$, though, as they do not require representability.

- *Non-Cartesian squares.* In [29, Th. 7.10] we proved some commutative diagrams of 1-morphisms of configuration moduli stacks are Cartesian. In the triangulated category case the corresponding diagrams may be commutative, but not Cartesian. So the multiplication $*$ on $\mathrm{CF}, \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ may no longer be associative, which spoils most of [30, 31] and this paper.

  The reason is the 'nonfunctoriality of the cone' in triangulated categories, discussed by Gelfand and Manin [18, p. 245]. If $\phi : X \to Y$ is a morphism in an abelian category $\mathcal{A}$, the kernel and cokernel of ϕ are determined up to canonical isomorphism. But in a triangulated category $\mathcal{T}$ the cone on ϕ is determined up to isomorphism, but not canonically. This means that the triangulated category versions of operations on configurations such as substitution [29, Def. 5.7] will be defined up to isomorphism, but not canonically, which will invalidate the proof of [29, Th. 7.10].

  I expect the right solution is to change the definition of triangulated category, perhaps following Neeman [36] or Bondal and Kapranov [6]. Neeman's proposal includes as data a category $\mathcal{S}$ of triangles in $\mathcal{T}$, but the morphisms in $\mathcal{S}$ are different from morphisms of triangles in $\mathcal{T}$, which restores the functoriality of the cone. If in defining configurations we took the distinguished triangles to be objects in $\mathcal{S}$, and their morphisms to include morphisms in $\mathcal{S}$, I hope the relevant squares will be Cartesian.

  In the dg-category approach of Toën and Vaquié [43] this problem may not occur. In particular, Toën [42] defines *derived Hall algebras* for certain dg-categories $\mathcal{T}$, which are associative. This is an analogue of associativity $*$ on $\underline{\mathrm{SF}}(\mathfrak{Obj}_{\mathcal{A}})$, and suggests that the programme ought to work.

- *Defining stability conditions on triangulated categories.* Here we can use Bridgeland's wonderful paper [9], and its extension by Gorodentscev et al. [19], which combines Bridgeland's idea with Rudakov's definition for abelian categories [39]. Since our stability conditions are based on [39], the modifications for our framework are straightforward.

- *Invariants and changing stability conditions.* Given a triangulated category $\mathcal{T}$ and a Bridgeland stability condition τ on $\mathcal{T}$, there is a t-structure on $\mathcal{T}$ whose heart is an abelian category $\mathcal{A}$, and τ determines a slope function μ on $\mathcal{A}$, and $X \in \mathcal{T}$ is τ-semistable if and only if $X = Y[n]$ for some μ-semistable $Y \in \mathcal{A}$ and $n \in \mathbb{Z}$. Here $\mathcal{A}$ is not unique, but two $\mathcal{A}, \mathcal{A}'$ coming from $\mathcal{T}, \tau$ are related by tilting. Thus invariants counting τ-semistable objects in $\mathcal{T}$ are essentially the same as invariants counting μ-semistable objects in $\mathcal{A}$, which we have studied in this paper.

  We can also reduce changing stability conditions in triangulated categories to the abelian category case, in the following way. Let $\tau, \tilde{\tau}$ be stability conditions on $\mathcal{T}$ which are 'sufficiently close' in some sense. Then the author expects that we can find a third stability condition $\hat{\tau}$ on $\mathcal{T}$ and two t-structures on $\mathcal{T}$ with hearts $\mathcal{A}_1, \mathcal{A}_2$, such that the first t-structure is compatible with $\tau, \hat{\tau}$, which arise from slope functions $\mu_1, \hat{\mu}_1$ on $\mathcal{A}_1$,

and the second t-structure is compatible with $\hat\tau, \tilde\tau$, which arise from slope functions $\hat\mu_2, \tilde\mu_2$ on $\mathcal{A}_2$.

As both t-structures are compatible with $\hat\tau$, $\mathcal{A}_1, \mathcal{A}_2$ are related by tilting, and $\hat\mu_1$-semistable objects in $\mathcal{A}_1$ are essentially the same as $\hat\mu_2$-semistable objects in $\mathcal{A}_2$. Thus transforming between invariants counting τ- and $\tilde\tau$-semistable objects in $\mathcal{T}$ is equivalent to transforming between invariants counting μ_1- and $\hat\mu_1$-semistable objects in the abelian category $\mathcal{A}_1$, and then transforming between invariants counting $\hat\mu_2$- and $\tilde\mu_2$-semistable objects in the abelian category $\mathcal{A}_2$, which we already know how to do.

Because of this reduction to the abelian category case, the author is confident that there should be a well-behaved theory of invariants counting τ-semistable objects in triangulated categories, despite all the problems described above.

In §6.4 we proposed that invariants counting γ-semistable sheaves on a $K3$ surface should be combined in a holomorphic *generating function* (122). This had a simple form because for $K3$ surfaces the invariants $\bar J^\alpha(\gamma)$ and conjectured invariants $\hat J^\alpha$ were independent of the choice of weak stability condition. Our next problem concerns the 'right' way to form generating functions when the invariants are not independent of weak stability condition.

Problem 7.2. (a) *Let $\mathcal{A}, K(\mathcal{A}), \mathfrak{F}_{\mathcal{A}}$ be one of the examples defined using quivers in* [29, Ex.s 10.5–10.9]*, such as $\mathcal{A} = \text{mod-}\mathbb{K}Q$. Let $c, r : K(\mathcal{A}) \to \mathbb{R}$ be group homomorphisms with $r(\alpha) > 0$ for all $\alpha \in C(\mathcal{A})$. Define a slope function $\mu : C(\mathcal{A}) \to \mathbb{R}$ by $\mu(\alpha) = c(\alpha)/r(\alpha)$. Then $(\mu, \mathbb{R}, \leqslant)$ is a permissible stability condition by* [31, Ex. 4.14]*. Define $Z : K(\mathcal{A}) \to \mathbb{C}$ by $Z(\alpha) = -c(\alpha) + ir(\alpha)$. Then Z determines μ, and the family of Z arising from slope functions μ in this way is an open subset U in the complex vector space* $\mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$.

Form invariants $I_{\mathrm{ss}}(I, \preceq, \kappa, \mu)$ or $I^\alpha_{\mathrm{ss}}(\mu)^\Lambda$ or $J^\alpha(\mu)^{\Lambda^\circ}$ or $J^\alpha(\mu)^\Omega$ counting μ-semistable objects or configurations in $\mathcal{A}$, as in §6. Find natural ways to combine these invariants in generating functions $F : U \to V$, where V is $\mathbb{C}, \Lambda \otimes_{\mathbb{Q}} \mathbb{C}, \Lambda^\circ \otimes_{\mathbb{Q}} \mathbb{C}$ or $\Omega \otimes_{\mathbb{Q}} \mathbb{C}$, and $F(Z)$ depends on Z and the values of the invariants for the stability condition $(\mu, \mathbb{R}, \leqslant)$ depending on Z. Is F holomorphic? If it is defined by an infinite sum, does it converge? Note that the $I_{\mathrm{ss}}(I, \preceq, \kappa, \mu), \ldots$ change discontinuously with μ and Z according to the transformation laws (82)*,* (97)*,* (98) *or* (129)*. Can we make F continuous under these changes?*

(b) *Let $\mathcal{T}$ be a triangulated category and* $\mathrm{Stab}(\mathcal{T})$ *the moduli space of Bridgeland stability conditions* [9] *on $\mathcal{T}$. Each $\tau \in \mathrm{Stab}(\mathcal{T})$ has an associated central charge $Z : K(\mathcal{A}) \to \mathbb{C}$, and the map $\tau \mapsto Z$ induces a map* $\mathrm{Stab}(\mathcal{T}) \to \mathrm{Hom}(K(\mathcal{T}), \mathbb{C})$ *which is a local diffeomorphism, and gives* $\mathrm{Stab}(\mathcal{T})$ *the structure of a complex manifold. When $\mathcal{T} = D^b(\mathcal{A})$ for $\mathcal{A}$ as in* (a)*, each μ, Z in* (a) *determines a unique $\tau \in \mathrm{Stab}(\mathcal{T})$ with the same Z, such that $X \in \mathcal{A} \subset D^b(\mathcal{A})$ is μ-semistable in $\mathcal{A}$ if and only if it is τ-semistable in $\mathcal{T}$.*

Suppose we can solve Problem 7.1 and have a good theory of invariants $I^\alpha_{\mathrm{ss}}(\tau), \ldots$ counting τ-semistable objects in $\mathcal{T}$, for $\tau \in$ $\mathrm{Stab}(\mathcal{T})$*. As in* (a)*, try*

to encode these in holomorphic generating functions $F : \mathrm{Stab}(\mathcal{T}) \to V$*. Can we make* F *continuous when the* $I_{\rm ss}^\alpha(\tau)$ *change discontinuously with* τ*?*

(c) *Now let* $\mathcal{T} = D^b(\mathrm{coh}(P))$ *for* P *a Calabi–Yau* 3*-fold. Can we form holomorphic generating functions* F *on* $\mathrm{Stab}(\mathcal{T})$ *encoding derived category versions of the* $J^\alpha(\tau)$ *of §6.5, or the* $\bar{DT}^\alpha(\tau)$ *of Conjecture 6.30? If so, does* F *encode some structure on* $\mathrm{Stab}(\mathcal{T})$ *important in String Theory and Mirror Symmetry; for instance, can we recover the stringy Kähler moduli space, the subset of* $\mathrm{Stab}(\mathcal{T})$ *corresponding to complex structures on the mirror Calabi–Yau* 3*-fold, from* F*?*

In a sequel [32], we present the author's attempt at solving this problem. Rather than a single generating function F, we define a family of generating functions F^α for all $\alpha \in C(\mathcal{A})$, or $\alpha \in K(\mathcal{T}) \setminus \{0\}$ in the triangulated case. For example, the invariants $J^\alpha(\tau)^\Omega$ of §6.5 are encoded in maps $F^\alpha : U$ or $\mathrm{Stab}(\mathcal{T}) \to \Omega \otimes_{\mathbb{Q}} \mathbb{C}$ of the form

$$\begin{aligned} F^\alpha(Z) = \sum_{\substack{n \geqslant 1,\ \alpha_1,\ldots,\alpha_n \in C(\mathcal{A}) \text{ or } K(\mathcal{T})\setminus\{0\}: \\ \alpha_1+\cdots+\alpha_n=\alpha,\ Z(\alpha_k)\neq 0 \text{ all } k}} & F_n\bigl(Z(\alpha_1),\ldots,Z(\alpha_n)\bigr) \prod_{i=1}^n J^{\alpha_i}(\mu)^\Omega \cdot \\ & \left[\frac{1}{2^{n-1}} \sum_{\substack{\text{connected, simply-connected digraphs } \Gamma: \\ \text{vertices } \{1,\ldots,n\},\ \text{edge } \overset{i}{\bullet} \to \overset{j}{\bullet} \text{ implies } i<j}} \prod_{\substack{\text{edges} \\ \overset{i}{\bullet} \to \overset{j}{\bullet} \\ \text{in } \Gamma}} \bar{\chi}(\alpha_i,\alpha_j) \right], \end{aligned} \tag{131}$$

where $F_n : (\mathbb{C}^\times)^n \to \mathbb{C}$ are some functions to be determined, and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

Supposing that the $J^\alpha(\tau)^\Omega$ transform according to (129), and neglecting issues to do with convergence of infinite sums, in [32] we show that there is an essentially *unique* family of functions F_n such that (131) yields F^α which are both continuous and holomorphic. These F_n are special functions of polylogarithm type, and are holomorphic at $(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n$ except along the real hypersurfaces $\arg(z_k) = \arg(z_{k+1})$ for $1 \leqslant k < n$, where F_n is discontinuous.

The point is that as we cross a real hypersurface in U or $\mathrm{Stab}(\mathcal{T})$ where $\arg(Z(\beta)) = \arg(Z(\gamma))$ for some $\beta, \gamma \in C(\mathcal{A})$ or $K(\mathcal{T}) \setminus \{0\}$, both the invariants $J^{\alpha_i}(\mu)^\Omega$ and the functions $F_n\bigl(Z(\alpha_1), \ldots, Z(\alpha_n)\bigr)$ in (131) can jump discontinuously. We arrange that these jumps exactly cancel out, so that F^α remains continuous. Very surprisingly, it turns out that with these choices of functions F_n the F^α satisfy the p.d.e.

$$\mathrm{d}F^\alpha(Z) = -\sum_{\beta,\gamma \in C(\mathcal{A}) \text{ or } K(\mathcal{T})\setminus\{0\}\,:\, \alpha=\beta+\gamma} \bar{\chi}(\beta,\gamma) F^\beta(Z) F^\gamma(Z) \frac{\mathrm{d}(Z(\beta))}{Z(\beta)}, \tag{132}$$

which can be interpreted as the flatness of an infinite-dimensional connection over U or $\mathrm{Stab}(\mathcal{T})$. This still leaves many questions unanswered, for example, mathematical questions on the convergence of the infinite sums (131) and (132), and physical questions on the interpretation of (131) and (132) in String Theory.

Here is our final problem.

Problem 7.3. *Redo* [28–31] *and this paper using instead of Artin stacks some other kind of stack, which at points* $[X]$ *includes the data* $\mathrm{Ext}^i(X,X)$ *for* $i>1$ *as part of its structure. In this framework try to generalize results in* [30, 31] *and this paper which work when* $\mathrm{Ext}^i(X,Y)=0$ *for all* $i>1$ *and* $X,Y\in\mathcal{A}$ *to the general case, in particular the morphisms to explicit algebras in* [30, §6].

Some good candidates for the appropriate notion of stack are the *D-stacks* of Toen and Vessozi [44] and the *dg-stacks* of Ciocan-Fontanine and Kapranov [13, §5]. Both of these papers explain relevant ideas in *derived algebraic geometry*.

The motivation behind this problem is that in constructing the algebra morphism Φ^Λ in [29, §6.2], if we could include the groups $\mathrm{Ext}^i(X,X)$ for $i>1$ appropriately then Φ^Λ would be an algebra morphism without assuming $\mathrm{Ext}^i(X,Y)=0$ for all X,Y and $i>1$. So we must work with a kind of stack including the data $\mathrm{Ext}^i(X,X)$ for $i>1$, and new 'derived stack function' spaces $\mathrm{SF}^{\mathrm{der}}(\mathfrak{Obj}_{\mathcal{A}})$.

There is a cost to this, though: the author expects Theorem 5.2 to fail for the spaces $\mathrm{SF}^{\mathrm{der}}(\mathfrak{Obj}_{\mathcal{A}})$, since in the proof of Theorem 5.11, the 1-morphism $\boldsymbol{\sigma}(\{1,\ldots,n\})$ is a 1-isomorphism of Artin stacks with a substack of $\mathfrak{Obj}_{\mathcal{A}}$, but it is probably *not* a 1-isomorphism of D-stacks or dg-stacks, as it may not preserve the data $\mathrm{Ext}^i(X,X)$ for $i>1$. So our material on change of stability condition will not work in $\mathrm{SF}^{\mathrm{der}}(\mathfrak{Obj}_{\mathcal{A}})$ in general.

However, in the situation of §6.4 when $\mathcal{A}=\mathrm{coh}(P)$ for K_P^{-1} nef, the author expects that at least equation (45) of Theorem 5.2 should hold in some suitable class of spaces $\mathrm{SF}^{\mathrm{der}}(\mathfrak{Obj}_{\mathcal{A}})$, as §6.4 works precisely by forcing $\mathrm{Ext}^i(X,Y)=0$ for $i>1$ and the relevant X,Y. Then we could interpret Theorem 6.21 as saying that we have an algebra morphism $\mathrm{SF}^{\mathrm{der}}(\mathfrak{Obj}_{\mathcal{A}})\rightarrow A(\mathcal{A},\Lambda,\chi)$, and the invariants $I_{\mathrm{ss}}^\alpha(\gamma)$ encode the restriction of this to a derived version $\bar{\mathcal{H}}_\gamma^{\mathrm{to,der}}$ of $\bar{\mathcal{H}}_\gamma^{\mathrm{to}}$, which is independent of stability condition $(\gamma,G_2,\leqslant)$. This would clear up a mystery about Theorem 6.21, that is, why we have invariants with multiplicative transformation laws but no underlying (Lie) algebra morphism.

References

[1] M.F. Atiyah and R. Bott, *The Yang–Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. London A308 (1982), 523–615.

[2] K. Behrend and A. Dhillon, *On the motive of the stack of bundles*, math.AG/0512640, 2005.

[3] D.J. Benson, *Representations and cohomomology: I*, Cambridge studies in advanced mathematics 30, Cambridge University Press, Cambridge, 1995.

[4] A. Bialynicki-Birula, *Some theorems on the actions of algebraic groups*, Annals of Math. 98 (1973), 480–497.

[5] E. Bifet, F. Ghione and M. Letizia, *On the Abel–Jacobi map for divisors of higher rank*, Math. Ann. 299 (1994), 641–672.

[6] A.I. Bondal and M.M. Kapranov, *Enhanced triangulated categories*, Math. USSR Sbornik 70 (1991), 93–107.

[7] R. Borcherds, *Automorphic forms on $O_{s+2,s}(\mathbb{R})$ and infinite products*, Invent. math. 120 (1995), 161–213.

[8] A. Borel, *Linear Algebraic Groups*, second edition, Graduate Texts in Math. 126, Springer-Verlag, New York, 1991.

[9] T. Bridgeland, *Stability conditions on triangulated categories*, math.AG/0212237, 2002. To appear in Annals of Mathematics.

[10] T. Bridgeland, *Stability conditions on $K3$ surfaces*, math.AG/0307164, 2003.

[11] T. Bridgeland, *Derived categories of coherent sheaves*, pages 563–582 in M. Sanz-Solé, J. Soria, J.L. Varona and J. Verdera, editors, *Proceedings of the International Congress of Mathematicians, Madrid, 2006*, vol. II, 2007. math.AG/0602129.

[12] *Anneaux de Chow et applications*, Séminaire C. Chevalley, 2e année, École Normale Supérieure, Secrétariat mathématique, Paris, 1958.

[13] I. Ciocan-Fontanine and M.M. Kapranov, *Derived Hilbert schemes*, J. A.M.S. 15 (2002), 787–815. math.AG/0005155.

[14] A. Craw, *An introduction to motivic integration*, pages 203–225 in M. Douglas, J. Gauntlett and M. Gross, editors, *Strings and Geometry*, Clay Math. Proc. 3, A.M.S., 2004. math.AG/9911179.

[15] U.V. Desale and S. Ramanan, *Poincaré polynomials of the variety of stable bundles*, Math. Ann. 216 (1975), 233–244.

[16] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford Mathematical Monographs, OUP, Oxford, 1990.

[17] S.K. Donaldson and R.P. Thomas, *Gauge Theory in Higher Dimensions*, chapter 3 in S.A. Huggett, L.J. Mason, K.P. Tod, S.T. Tsou and N.M.J. Woodhouse, editors, *The Geometric Universe*, Oxford University Press, Oxford, 1998.

[18] S.I. Gelfand and Y.I. Manin, *Methods of Homological Algebra*, second edition, Springer Monographs in Mathematics, Springer, Berlin, 2003.

[19] A. Gorodentscev, S. Kuleshov and A. Rudakov, *Stability data and t-structures on a triangulated category*, math.AG/0312442, 2003.

[20] T.L. Gómez, *Algebraic stacks*, Proc. Indian Acad. Sci. Math. Sci. 111 (2001), 1–31. math.AG/9911199.

[21] G. Harder and M.S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. 212 (1975), 215–248.

[22] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York, 1977.

[23] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Aspects of Mathematics E31, Vieweg, Braunschweig, 1997.

[24] V.A. Iskovskikh and I.R. Shafarevich, *Algebraic surfaces*, Part II of I.R. Shafarevich, editor, *Algebraic Geometry II*, Encylopaedia of Mathematical Sciences vol. 35, Springer-Verlag, Berlin, 1996.

[25] D.D. Joyce, *On counting special Lagrangian homology* 3*-spheres*, pages 125–151 in A.J. Berrick, M.C. Leung and X.W. Xu, editors, *Topology and Geometry: Commemorating SISTAG*, Contemporary Mathematics volume 314, A.M.S., 2002. hep-th/9907013.

[26] D.D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. V. Survey and applications*, J. Diff. Geom. 63 (2003), 279–347. math.DG/0303272.

[27] D.D. Joyce, *Constructible functions on Artin stacks*, J. London Math. Soc. 74 (2006), 583–606. math.AG/0403305.

[28] D.D. Joyce, *Motivic invariants of Artin stacks and 'stack functions'*, math.AG/0509722, 2005. To appear in the Oxford Quarterly Journal of Mathematics.

[29] D.D. Joyce, *Configurations in abelian categories. I. Basic properties and moduli stacks*, Advances in Math. 203 (2006), 194–255. math.AG/0312190.

[30] D.D. Joyce, *Configurations in abelian categories. II. Ringel–Hall algebras*, Advances in Math. 210 (2007), 635–706. math.AG/0503029.

[31] D.D. Joyce, *Configurations in abelian categories. III. Stability conditions and identities*, math.AG/0410267, version 5, 2007.

[32] D.D. Joyce, *Holomorphic generating functions for invariants counting coherent sheaves on Calabi–Yau* 3*-folds*, hep-th/0607039, 2006. To appear in Geometry and Topology.

[33] M. Kontsevich, *Homological Algebra of Mirror Symmetry*, in Proc. Int. Cong. Math. Zürich, 1994. alg-geom/9411018.

[34] A. Langer, *Semistable sheaves in positive characteristic*, Ann. Math. 159 (2004), 251–276.

[35] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergeb. der Math. und ihrer Grenzgebiete 39, Springer-Verlag, Berlin, 2000.

[36] A. Neeman, *Some new axioms for triangulated categories*, J. Algebra 139 (1991), 221–255.

[37] D.O. Orlov, *Equivalences of derived categories and $K3$ surfaces*, J. Math. Sci. (New York) 84 (1997), 1361–1381. alg-geom/9606006.

[38] M. Reineke, *The Harder–Narasimhan system in quantum groups and cohomology of quiver moduli*, Invent. math. 152 (2003), 349–368. math.QA/0204059.

[39] A. Rudakov, *Stability for an Abelian Category*, Journal of Algebra 197 (1997), 231–245.

[40] R.P. Thomas, *A holomorphic Casson invariant for Calabi–Yau 3-folds, and bundles on $K3$ fibrations*, J. Diff. Geom. 54 (2000), 367–438. math.AG/9806111.

[41] B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, math.QA/0408337, 2004.

[42] B. Toën, *Derived Hall algebras*, Duke Math. J. 135 (2006), 587–615. math.QA/0501343.

[43] B. Toën and M. Vaquié, *Moduli of objects in dg-categories*, math.AG/0503269, 2005.

[44] B. Toën and G. Vezzosi, *From HAG to DAG: derived moduli stacks*, pages 173–216 in J.P.C. Greenless, editor, *Axiomatic, enriched and motivic homotopy theory*, NATO Science ser. II vol. 131, Kluwer, Dordrecht, 2004. math.AG/0210407.

[45] K. Yoshioka, *The Betti numbers of the moduli space of stable sheaves of rank 2 on a ruled surface*, Math. Ann. 302 (1995), 519–540.

[46] K. Yoshioka, *Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface*, Int. J. Math. 7 (1996), 411–431.

[47] K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. 321 (2001), 817–884. math.AG/0009001.

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